# String field theory: Time evolution and T-duality

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A Thesis presented for the degree of
Doctor of Philosophy

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*We are indeed drifting into the arena of the unwell.*

– Marwood

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## Abstract

The time evolution operator of quantum field theory (Schrödinger functional) can be written in terms of particles moving on $\mathbb{S}^1/\mathbb{Z}_2$. By deriving the 'gluing property' which joins two propagators across fixed time surfaces, we show that the Feynman diagram expansion of the free Schrödinger functional is determined once we know the field propagator.

We generalise the gluing property to a new method of sewing string field propagators and construct the string field Schrödinger functional in terms of strings moving on $\mathbb{S}^1/\mathbb{Z}_2$. Timelike T-duality in string theory then appears as a large/small time symmetry of string field theory with an exchange of boundary states and string backgrounds. All of our arguments apply equally to the open and closed string.

The addition of interactions to quantum field theory bring no complication to our arguments, but modifications are required when the interaction is non-local. As as application of these methods we construct the interacting string field vacuum wave functional using a knowledge of the vacuum expectation values it must generate.

# Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, Durham UK. No part of this thesis has been submitted elsewhere for any other degree or qualification. Chapter one is review material and no claim of originality is made. The remaining chapters contain original material and material developed in collaboration with Prof. P. Mansfield, the majority of which is published in references [50], [51], [52], [60].

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The very many Feynman diagrams, figures and graphical equations in this thesis were drawn in 'Jaxodraw', created by D. Binosi and L. Theußl. See reference [1].

# Contents

# Chapter 1

# Introduction

String theory, like quantum mechanics, is discussed mostly in the formalism of first quantisation theory. The extended nature of the string makes calculations appear similar to those in field theory, since there are an infinite number of degrees of freedom representing the infinite number of possible vibrational modes of the string.

Although the theory generates its own interactions as represented by the Polyakov sum over topologies, this series is perturbative. A string field theory is needed to give a non–perturbative description of strings and give us access to information hidden from perturbation theory.

In this thesis we study the nature of time in string field theory. This second quantised theory sees spacetime as being built from curves, not points, which implies that our perceptions of time in particle theories may need to be altered.

To understand the issues involved it may be beneficial to contrast the theory of particles. Most particle physics is done in second quantisation. Rather than quantise a finite number of degrees of freedom (the position $x^\mu$ and momentum $p_\mu$ of a particle) we quantise an infinite number, namely $\phi(x^\mu)$, the field argument at position $x^\mu$. The field can be pictured as being much like the classical field of say, electromagnetism (in whichever way the reader chooses to visualise such things). It exists throughout spacetime and excitations of the field can appear locally as particles. The quantum field is not really the smooth field of classical theory, the uncertainty principle means that even the supposedly empty vacuum is in a state of constant flux, but as a mental picture the classical field should suffice.

The field is not restricted to describing single particle states and so is the natural framework in which to investigate the multi-particle state processes such as pair production and annihilation observed in nature.

The natural picture in which to investigate time evolution in quantum field theory is the Schrödinger representation. The field operator is diagonalised on the initial quantisation surface, the hyperplane at some fixed time, and arbitrary data is evolved through time from this surface by the action of the Schrödinger functional.

What happens when we go to string theory? The string field exists throughout spacetime, but its arguments are not spacetime points but rather one dimensional spacetime curves[1]. Excitations of the string field appear locally as strings. We wish to construct the Schrödinger functional for strings though it is not clear how to realise this using current approaches to string field theory.

For example, in Witten's open string field theory the time variable is associated with the midpoint of the timelike extension of the string $X^0(\pi/2)$, rather than being a global time for the whole string. It is unclear what this implies for our interpretation of time nor how a Schrödinger representation might be constructed. One possibility is to use the light-cone gauge but there are concerns in the literature over the usefulness of this theory in describing non-perturbative information (more on this later).

Our aim is to construct the time evolution operator of (bosonic) string field theory. The vacuum state wave functional will feature heavily in our particle calculations, so we will also construct the string field vacuum. Both time dependence and the nature of the string field vacuum are topics of current interest and we hope to be able to add to the literature on them.

Due to the difficulties of working with known Lagrangian formalisms, in this thesis we find diagrammatic methods of examining second quantised string theory using properties of first quantisation. This approach is inspired by analogous results in field theory.

The thesis is laid out as follows. We begin by reviewing the Schrödinger represen-

[1] Unlike particle theories, it has long been known that outside of the light-cone gauge the string field must also be a functional of the reparametrisation ghosts of string theory.

tation of scalar quantum field theory (scalar fields will be our toy model throughout). In field theory the Schrödinger picture is regularly abandoned in favour of Fock space realisations based on creation and annihilation operators. In quantum mechanics it is more familiar and closely related to the conceptual interpretation of the theory, so we review this material before we describe the field theory representation to help solidify ideas.

The remainder of this chapter is given over to a review of the functional approach to string theory and then string field theory. We summarise the differences between the functional and canonical quantisation of the string, describe how to compute the functional integrals, how the dimension of spacetime arises, gauge fixing and the calculation of on-shell scattering amplitudes. We then describe the need for a string field theory, open problems, and two of the most successful approaches, namely the light-cone gauge and Witten's cubic theory.

In Chapter 2 we define the Schrödinger functional and vacuum functional in scalar field theory, their sum over field histories descriptions as given by Symanzik and their Feynman diagram expansion. We show that these functionals can be expressed in terms of particles moving on $\mathbb{R}^D$ times a discrete symmetrisation of the time direction using the sum over paths representation of the field propagator. Since the sum over paths has a natural generalisation to the Polyakov integral for strings, it is suggested that we can construct the string field Schrödinger functional from first quantised strings by analogy.

To realise this we derive the 'gluing property'. This is a property of the free space scalar field propagator fundamental to second quantisation, but derived in first quantisation. It is a method of gluing together reparametrisation invariant propagators and diagrams, including their moduli spaces, appropriate to the Schrödinger representation. Using this property we show that time dependence as defined through the action of the Schrödinger functional can be described graphically; the Schrödinger functional is characterised solely by the gluing property and its Feynman diagram expansion.

Our conclusion is that once we have the free field two-point function (the free propagator) the Schrödinger functional and vacuum wave functional are determined, provided that the gluing properties hold. The string field propagator is well known, as we review at the beginning of Chapter 3. We then prove that the gluing property holds as a method of sewing worldsheets by using a particular BRST constraint to quantise string theory. This approach is unconventional, so we verify our gluing rules using the cancellation of the Weyl anomaly. The Schrödinger functional and vacuum wave functional for both open and closed free bosonic strings are then described.

In Chapter 4 we introduce interactions into our theories. We calculate a three field correlation function in $\phi^3$ theory using our graphical arguments to describe how the Schrödinger representation relates to standard covariant results. The interaction vertex in string field theory is cubic but non-local. When we try to extend our methods to non-locally interacting quantum field theories the functional descriptions fail, for reasons we explain. We describe an alternative method of constructing the functionals which applies to both local and non-locally interacting field theories. We close the chapter by using these methods to describe the vacuum wave functional of interacting string field theories. Our calculation does not require the choice of a specific interaction vertex and again applies to both open and closed strings.

In Chapter 5 we describe the Schrödinger functional in the field momentum representation. In this representation the functional is built from particles moving on an orbifolded time direction. Naturally this leads to T-duality in the string field Schrödinger functional. We show that timelike T-duality of string theory becomes a large/ small time duality of string field theory, where evolution through time $t$ is exchanged with evolution through time $1/t$ and with an interchange of string fields and backgrounds. Finally we give our conclusions.

The appendices contain some proofs of gluing properties for string field theory and extensions of some quantum field theory results to anti de Sitter spacetimes.

## 1.1 The Schrödinger representation of quantum mechanics

Non-relativistic quantum mechanics is invariably described in the Schrödinger representation, since it makes contact with physical observables. We summarise this here as an aid to understanding the analogous, but less familiar, representation of quantum field theory.

The particle of mass $m$ moving in $D$ space dimensions with a potential $V(\mathbf{x})$ has action

$$S[\mathbf{x}] = \int \mathrm{d}t \, \frac{m\dot{\mathbf{x}}^2}{2} - V(\mathbf{x}),$$

where $t$ is time and a dot is a derivative with respect to time. We define the conjugate momentum

$$\mathbf{p}_i \equiv \frac{\partial L}{\partial \dot{\mathbf{x}}^i} = m\dot{\mathbf{x}}^i,$$

and to quantise promote $\mathbf{x}$ and $\mathbf{p}$ to operators obeying the equal time canonical commutation relations

$$[\hat{\mathbf{x}}^i(t), \hat{\mathbf{p}}_j(t)] = i\hbar\, \delta^i_j.$$

Quantum states $|\,\psi\,\rangle$ are represented as wave functions by picking a basis for the state space. In the Schrödinger representation we take the basis to be the set of position eigenstates $\langle\,\mathbf{x}\,|$ such that

$$\langle\,\mathbf{x}\,|\hat{\mathbf{x}}^i(0) = \mathbf{x}^i\langle\,\mathbf{x}\,|,$$

i.e. we diagonalise the position operator at some given time. State wave functions are then $\psi(\mathbf{x}, t) = \langle\,\mathbf{x}\,|\,\psi\rangle$. The basis has the properties

$$\begin{aligned} \int \mathrm{d}^D\mathbf{x}\,|\,\mathbf{x}\,\rangle\langle\,\mathbf{x}\,| &= 1, \\ \langle\,\mathbf{x}\,|\,\mathbf{y}\rangle &= \delta^D(\mathbf{x} - \mathbf{y}) \end{aligned}$$

The dependence of the states on $\mathbf{x}$ can be made explicit by writing

$$\langle\,\mathbf{x}\,| = \langle\,D\,|e^{i\hat{\mathbf{p}}.\mathbf{x}/\hbar}$$

where $\langle\,D\,|$ is the state annihilated by $\hat{\mathbf{x}}$, i.e. the state $\mathbf{x} = 0$. The momentum operator, therefore, acts as a derivative on wave functionals,

$$\langle\,\mathbf{x}\,|\hat{\mathbf{p}}_i = -i\hbar\frac{\partial}{\partial \mathbf{x}^i}\langle\,\mathbf{x}\,|.$$

The time dependence of the state is given by the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\,\psi\,\rangle = \hat{H}|\,\psi\,\rangle$$

which, using the definition of the momentum operator above, can be written as the differential equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = \left( -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \mathbf{x}^2} + V(\mathbf{x})\right)\psi(\mathbf{x},t).$$

The probabilistic interpretation of the wavefunction is that

$$\mathrm{d}^D\mathbf{x}\,\left|\psi(\mathbf{x},t)\right|^2$$

is the probability of measuring the position of the particle in state $\psi$ and finding it in the volume $\mathrm{d}^D\mathbf{x}$. This gives the wavefunction normalisation condition

$$\int \mathrm{d}^D\mathbf{x}\,\left|\psi(\mathbf{x},t)\right|^2 = 1.$$

## 1.2 The Schrödinger representation of quantum field theory

In quantum field theory the common approach to canonical quantisation is to use the Heisenberg representation where the operators are time dependent and states are time independent. The field operator is expanded in Fourier modes, and a Fock space is built on which these modes act as creators and annihilators of particles.

We will instead work in the Schrödinger picture where the states are time dependent and operators are time independent. This is summarised below and the description is parallel to that of the Schrödinger picture of quantum mechanics.

Note that the Schrödinger representation of field theory is not manifestly Lorentz covariant, since we have singled out time as a special direction[2]. This is one of the reasons why, historically, the Schrödinger representation was the less favoured approach. Symmetries were essential in the development of, for example, renormalisation theory, and comparison with experiment is most suited to the Fock space approach. We will see in the next chapter that, by the same token, the Schrödinger representation is ideally suited to studying time evolution.

[2]For the previous case this was not an issue since there was no Lorentz symmetry to maintain.

A scalar field in $D+1$ dimensional Minkowski spacetime, signature $(+,-,-\ldots)$, has action

$$S[\phi]=\int \mathrm{d}^D\mathbf{x}\,\mathrm{d}t\;\frac{1}{2}\left(\partial_t\phi\partial_t\phi-\nabla\phi\cdot\nabla\phi\right)-V(\phi)$$

where $V$ contains mass and interaction terms. To quantise, the field $\phi$ and its conjugate momentum $\pi$,

$$\pi\equiv\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=\dot{\phi},$$

are promoted to operators obeying the equal time commutation relation

$$[\hat{\phi}(\mathbf{x},t),\hat{\pi}(\mathbf{y},t)]=i\hbar\,\delta\left(\mathbf{x}-\mathbf{y}\right).$$

The canonical commutation algebra is represented by diagonalising the field operator on the quantisation surface $t=0$. That is, a basis for the state space is given by

$$\langle\,\phi\,|\hat{\phi}(\mathbf{x},0)=\phi(\mathbf{x})\langle\,\phi\,|$$

with the properties

$$\begin{aligned}\int\mathcal{D}\phi\;|\,\phi\,\rangle\langle\,\phi\,| &= 1,\\ \langle\,\phi\,|\,\tilde{\phi}\rangle &= \delta[\phi-\tilde{\phi}].\end{aligned}$$

Using the canonical commutation relations the dependence on the field of these states is made explicit by writing

$$\langle\,\phi\,|=\langle\,D\,|\exp\left(\frac{i}{\hbar}\int\mathrm{d}^D\mathbf{x}\;\phi(\mathbf{x})\hat{\pi}(\mathbf{x},0)\right)$$

where the Dirichlet state $\langle\,D\,|$ is annihilated by $\hat{\phi}(\mathbf{x},0)$. The momentum operator acts as a functional derivative,

$$\langle\,\phi\,|\hat{\pi}(\mathbf{x})=-i\hbar\frac{\delta}{\delta\phi(\mathbf{x})}\langle\,\phi\,|.$$

A quantum state $|\,\Psi\,\rangle$ is represented by a functional of the field and depends explicitly on the time,

$$\langle\,\phi\,|\,\Psi\rangle=\Psi\left[\phi(\mathbf{x}),t\right],$$

the dependence on which is governed by the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\,\Psi\,\rangle=\hat{H}|\,\Psi\,\rangle.$$

This can be written as the functional differential equation

$$i\hbar\frac{\partial}{\partial t}\Psi[\phi(\mathbf{x}),t] = \int \mathrm{d}^D\mathbf{x}\,\left(-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\phi^2}+\frac{1}{2}\nabla\phi\cdot\nabla\phi+V(\phi)\right)\Psi[\phi(\mathbf{x}),t].$$

The probabilistic interpretation is that the probability of a measurement of the system in the state $\Psi$ will find the field within the set of configurations defined by $\mathcal{D}\phi$ is

$$\mathcal{D}\phi\left|\Psi[\phi]\right|^2,$$

which again gives a normalisation condition,

$$\int \mathcal{D}\phi\left|\Psi[\phi]\right|^2 = 1.$$

## 1.3 Canonical string theory

The first quantised string action is the generalisation of the relativistic particle action to a one dimensional object. The particle of mass $m$ has action equal to the mass multiplied by the length of the particle worldline,

$$S_{\text{part}} = m\int \mathrm{d}s = m\int \mathrm{d}\xi\,\sqrt{\dot{x}^\mu(\xi)\dot{x}_\mu(\xi)},$$

where $\xi$ is a parameter along the worldline. The Feynman prescription for calculating a quantum transition amplitude is to sum over all possible paths the particle can take weighted with the exponential of $i$ times the action. The difficulties of using this action are two-fold. First there is an enormous over counting of equivalent spacetime paths due to the reparametrisation invariance of the action ($x^\mu(\xi)$ and $x^\mu(f(\xi))$ are the same path), and secondly no-one knows how to do such integrals containing exponentials of square roots.

The solution to these problems is to introduce a metric $g$ on the worldline and use instead the action [2]

$$S_g = \frac{1}{2}\int \mathrm{d}\xi\sqrt{g}\left(m^2+\dot{x}^\mu(\xi)g^{-1}\dot{x}_\mu(\xi)\right)$$

(which is also suitable for $m=0$ particles). The equations of motion of the metric are $\dot{x}^2-m^2g=0$, and substituting this back into $S_g$ recovers the original action $S_{\text{part}}$, so these actions are classically equivalent. Under a reparametrisation, $\xi \to \xi'(\xi)$ the metric $g$ transforms as a (0,2) tensor,

$$g(\xi) = \left(\frac{d\xi'}{d\xi}\right)^2 g(\xi')$$

and the action $S_g$ is easily verified to be reparametrisation invariant. We now exploit this reparametrisation invariance and gauge fix the metric $g$ to something useful, for example $g = 1/m^2$. This gives us an action quadratic in $x^\mu$, which is much more workable, and gives a familiar expression for the canonical momenta, $p_\mu = m\dot{x}_\mu$. The equations of motion of the particle are now $\ddot{x}^\mu = 0$ so the particles move along straight lines, the geodesics of flat space.

However, when working with the gauge fixed action we must remember to impose the constraint coming from the equations of motion of $g$. In our gauge this is $\dot{x}^2 - 1 = 0$, which, using the definition of $p_\mu$, is the familiar mass-shell condition $p^2 = m^2$.

Canonical quantisation of the bosonic string is the repetition of these arguments for a one dimensional object. The value of the particle action is given by the length of the spacetime path a particle takes, so for a string we expect the Nambu-Goto action [3], [4], [5]

$$\frac{1}{2\pi\alpha'}\int d^2\sigma\,\sqrt{\det \partial_a X^\mu(\sigma)\partial_b X_\mu(\sigma)} \tag{1.1}$$

equal to the area of the spacetime region swept out by the string as it propagates. $2\pi\alpha'$ is the inverse string tension, $\sigma^a = (\tau, \sigma)$ is a two component vector parameterising a two dimensional worldsheet and the determinant is that of the two by two matrix of partial derivatives $\partial_a \equiv \partial/\partial\sigma^a$. $X^\mu(\sigma)$ are a set of $D+1$ spacetime curves giving the location of the string in $D+1$ dimensional spacetime.

Again, this action is difficult to work with because of the square root, so we introduce a metric $g_{ab}(\sigma)$ on the worldsheet giving us the Polyakov action [6], [2]

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\,\partial_a X^\mu(\sigma)g^{ab}(\sigma)\partial_b X_\mu(\sigma). \tag{1.2}$$

Integrating out $g_{ab}$ returns the Nambu-Goto action. The Polyakov action is invariant under reparametrisations and also, uniquely because we have a two dimensional worldsheet, under a Weyl scaling of the metric $g_{ab} \to e^{\rho(\sigma)}g_{ab}$ (which leaves $X^\mu$

invariant). This is called a conformal symmetry since it changes distance but not angles. $\rho$ is called the Liouville mode.

We can now gauge fix the metric as we did for the particle, but the details depend very much on the topology of the worldsheet. The string may be open or closed. For a free open (closed) string the parameter domain of $\sigma^a$ is unrestricted, so the worldsheet has the topology of a disk (sphere). It is a theorem of Gauss that on the disk (sphere) any metric can be transformed to any other using only a reparametrisation and a Weyl scaling. Therefore we can choose co-ordinates (gauge fix) such that

$$\sqrt{g}g^{ab} = \eta^{ab}, \tag{1.3}$$

which removes the metric degrees of freedom from the action. This is called the conformal gauge choice. Though we will lose manifest reparametrisation invariance this approach is explicitly Weyl invariant, provided we do not reintroduce the metric[3].

The gauge fixed action is

$$S_{\mathrm{gf}} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \partial_a X^\mu(\sigma)\partial^a X_\mu(\sigma), \tag{1.4}$$

and the equations of motion for $X^\mu$ are

$$\partial_a\partial^a X^\mu = 0 \tag{1.5}$$

with Neumann boundary conditions for the open string and periodic conditions on the closed string. As before we must remember to impose the 'missing' Euler-Lagrange constraint,

$$T_{ab}(X) \equiv \frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{ab}}\bigg|_{\sqrt{g}g^{ab}=\eta^{ab}} = 0 \tag{1.6}$$

which is Fourier expanded to give the set of Virasoro constraints imposing the physical state conditions. Using this and the equations of motion, the string spectrum is easily determined. The string, and therefore the Virasoro constraints, can be expanded in Fourier modes which are promoted to creation/ annihilation operators at quantisation. A state is described by a collection of the creation modes acting on a Fock vacuum. The physical state conditions impose restrictions on the allowed

[3]As may be required by renormalisation, for example.

combinations of those modes. For example, the lowest mass open string state is the famous tachyon, which has no oscillation modes excited but carries a momentum. The Virasoro constraints imply that the mass squared of this state is negative.

However, upon closer inspection it is found that the theory suffers from the presence of negative norm states, which give us nonsensical negative probabilities for physical processes. The Virasoro constraints are sufficient to eliminate these states provided that the dimension of spacetime is $D+1=26$.

Interactions occur when strings split or join. Scattering amplitudes are organised by worldsheet topology, which is the analogue of the loop expansion in quantum field theory. One difficulty of working in the canonical approach is that to calculate terms of order higher than tree level, one must begin with the operator formalism appropriate to topologies other than the disk and once again ensure that ghosts do not contribute to the loops, both of which are challenging. The functional approach of Polyakov is much better suited to this task, as we now describe.

## 1.4 Functional string theory

Polyakov's approach was to make explicit the reparametrisation invariance of all calculations at the expense of losing explicit Weyl invariance. This is the opposite of the canonical approach, where we gauge fixed from the outset. The starting point is the functional integral

$$\sum_{\chi} \beta^{-\chi} \int \mathcal{D}(X,g) \mathcal{N} W_1 \dots W_n e^{iS[g,X]}. \tag{1.7}$$

$\mathcal{N}$ is a normalisation constant,

$$\mathcal{N} = \frac{1}{\mathrm{Vol}(\mathrm{Diff} \times \mathrm{Weyl})}, \tag{1.8}$$

to regulate the over-counting of equivalent worldsheets. The $W_i$ are insertions we will return to when we discuss scattering. $\beta$ is the string coupling constant and $\chi$ is the Euler characteristic of the manifold, so the sum represents a genus expansion. To properly define the integrals we must define measures on the spaces of $X^\mu$ and $g_{ab}$, maintaining reparametrisation invariance.

We begin with the $X^\mu$ integrals. To understand the origin of the measure, consider for a moment a finite dimensional integral over a volume of 3–dimensional space. An infinitesimal change in volume can be written as the scalar triple product $\delta v = \delta_1\mathbf{x}.\delta_2\mathbf{x} \wedge \delta_3\mathbf{x}$ in terms of small displacements $\delta_i\mathbf{x}$ (this is the volume of the parallelepiped of sides $\delta_i\mathbf{x}$). This can be written as the determinant of a matrix,

$$\delta v = \det \delta_a x^b,$$

which implies, squaring the volume element and taking the determinant of the multiplied matrices, rather than the product of the determinants,

$$\delta v = \sqrt{\det(\delta_a\mathbf{x}, \delta_b\mathbf{x})}$$

which is written in terms of the inner product ( , ). We note that this formula can be applied to arbitrary dimensional space and, in passing, explains the form of the Nambu-Goto action (1.1). The trick is now to choose the variations to be defined as variations of the co-ordinates $a_j$ in a basis of vectors $\mathbf{e}_j$, so that

$$\delta_j\mathbf{x} = \delta a_j \mathbf{e}_j \quad \textit{where no sum is implied,}$$

which gives us

$$\delta v = \prod_k \delta a_k \sqrt{\det(\mathbf{e}_i, \mathbf{e}_j)}.$$

In the limit we replace '$\delta$' with 'd' and this defines the volume measure in terms of the inner product ( , ). This is generalisable to infinite dimensional spaces. In the Polyakov integral we wish to integrate over the volume of the space of functions $X^\mu$ (satisfying certain boundary conditions). We find a suitable basis of functions $e_n(\sigma^c)$, such that we can write

$$X^\mu(\sigma^c) = \sum_n a_n^\mu e_n(\sigma^c)$$

and then a basis of variations of $X^\mu$ is given by

$$(\delta a_n^\mu) e_n(\sigma^c)$$

for each $\mu$, for each $n$, in terms of variations of the coefficients $\delta a_n^\mu$. Following the same arguments as above the measure on this space is

$$\mathcal{D}X = \prod_{n,\mu} \mathrm{d}a_n^\mu \sqrt{\mathrm{Det}\,(e_r(\sigma), e_s(\sigma))}.$$

With this basis the string action becomes a function of the $a_n^\mu$ and we now have an infinite product of ordinary Riemann integrals over the coefficients $a_n^\mu$. To complete the definition of the measure we need to identify the inner product and hence the determinant. The inner product

$$(\delta X, \delta X) \equiv \|\delta X^2\| = \int \mathrm{d}^2\sigma\sqrt{g}\; \delta X^\mu \delta X_\mu \tag{1.9}$$

is reparametrisation invariant since the $X^\mu$ are worldsheet scalars, but is not Weyl invariant so nor will be the measure. Using integration by parts we can write the Polyakov action as

$$S = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{g}\; X^\mu \Delta X_\mu \tag{1.10}$$

up to classical pieces, where $\Delta$ is the worldsheet scalar Laplacian,

$$\Delta = -\frac{1}{\sqrt{g}}\partial_a\sqrt{g}g^{ab}\partial_b. \tag{1.11}$$

We choose the $e_n$ to be a basis of eigenvectors of the Laplacian. Since $\Delta$ is Hermitian with respect to the inner product we have defined we can choose its eigenfunctions to be orthonormal – the determinant in the measure becomes unity in this case and we are left with an infinite number of readily computable Gaussian integrals. When we perform these integrals we obtain the determinant (the infinite product of eigenvalues) of the operator $\Delta$, to the power minus one half .

However, this determinant is not Weyl invariant. Although this was classically a symmetry of the action, it is broken when we perform the functional integral – i.e. when we quantise the theory. This is the Weyl anomaly. This is not quite the whole story. Not only must we regulate the determinant, we have an integral over $g_{ab}$ to perform and it is possible that this will cancel the dependence on the Liouville mode (this is of course precisely what will happen if $D + 1 = 26$, as we will see).

The only complication to the $X^\mu$ integral is when the manifold and boundary conditions permit $\Delta$ to have a zero mode $e_0(\sigma)$, which must be a constant. The integral over $X^\mu$ then contains an integral over the coefficient of this zero mode which diverges. It contains information which must be separated from the divergence. The normalisation condition gives the value of the zero mode

$$(e_0, e_0) = 1 \implies e_0 = \left(\int \mathrm{d}^2\sigma\sqrt{g}\right)^{-1/2}$$

Now call the centroid of the string $x^\mu$ (the 'centre of mass' of the worldsheet),

$$x^\mu \equiv \frac{1}{\int d^2\sigma\sqrt{g}} \int d^2\sigma\sqrt{g}\, X^\mu = \frac{1}{\int d^2\sigma\sqrt{g}} (X^\mu, 1) = \frac{1}{\left(\int d^2\sigma\sqrt{g}\right)^{1/2}}\, a_0^\mu .$$

The divergent integral can therefore be written in terms of an integral over the centroid

$$\int da_0^\mu = \left(\int d^2\sigma\sqrt{g}\right)^{(D+1)/2} \int dx^\mu .$$

The final term is simply the volume of space, which we factor out. The final result of the $X$ integrations is

$$\left(\frac{\int d^2\sigma\sqrt{g}}{\mathrm{Det}'\Delta}\right)^{(D+1)/2} \tag{1.12}$$

possibly multiplied by the exponential of any classical piece. The numerator in the above appears only if we have a zero mode.

The integral over the metric follows similar lines. For the free string, an arbitrary variation of the metric can be written in terms of infinitesimal Weyl scalings $\delta\rho$ and reparametrisation $\delta\xi^a$ as

$$\delta g_{ab} = \delta\rho\, g_{ab} + \nabla_{(a}\delta\xi_{b)} . \tag{1.13}$$

There are in general additional parameters describing the metric (moduli) but we will return to these later. We need a reparametrisation invariant inner product on such variations. There are two possible choices,

$$(\delta_1 g, \delta_2 g) = A \int d^2\sigma\sqrt{g}\; \delta_1 g_{ab}\delta_2 g_{rs} g^{ar} g^{bs} + B \int d^2\sigma\sqrt{g}\; \delta_1 g_{ab}\delta_2 g_{rs} g^{ab} g^{rs} . \tag{1.14}$$

Once again, neither of these are Weyl invariant. Also, the two types of variation of the metric are not orthogonal with respect to either inner product (there are reparametrisations which appear as Weyl scalings of the metric) so we cannot proceed to define the measure as we did for the integral over $X$. We are looking for a functional determinant for this definition, and a determinant is unaffected by row and column operations on its matrix. In the infinite dimensional case, this tells us we can shift the piece of $\nabla_{(a}\delta\xi_{b)}$ which generates Weyl scalings into a redefinition of the Liouville mode $\delta\rho$. We write a variation of the metric as

$$\delta g_{ab} = \delta\rho\, g_{ab} + P(\delta\xi)_{ab} \tag{1.15}$$

where $P$ maps vectors into traceless, symmetric covariant tensors,

$$P(\delta\xi)_{ab} = \nabla_{(a}\delta\xi_{b)} - g_{ab}\nabla_c\delta\xi^c. \tag{1.16}$$

The two types of variation are now orthogonal, as we see if we insert this representation for $\delta g$ into (1.14),

$$(\delta_1 g, \delta_2 g) = (2A + 4B)\int \mathrm{d}^2\sigma\sqrt{g}\ \delta_1\rho\delta_2\rho + A\int \mathrm{d}^2\sigma\sqrt{g}\ \delta_1 P(\delta_1\xi)_{ar}g^{ab}g^{rs}P(\delta_2\xi)_{bs}.$$

After an integration by parts the second term becomes

$$A\int \mathrm{d}^2\sigma\sqrt{g}\ g_{ab}\delta_1\xi^a(P^\dagger P(\delta_2\xi))^b,$$

where $P^\dagger(b_{rs})^a = -2\nabla_b b^{ab}$. We can now define part of the measure. Expanding the reparametrisation in a basis of eigenfunctions of $P^\dagger P$,

$$\xi^a = \sum_N b_N U_N^a(\sigma)$$

defines the functional measure over reparametrisations

$$\int \mathcal{D}g = \int \mathcal{D}\rho \prod_N \mathrm{d}b_N \sqrt{\mathrm{Det}\,(U_N, P^\dagger P U_M)}.$$

We have been deliberately obscure about the nature of $\mathcal{D}\rho$ for reasons which will become apparent. Note that there will be in general be a finite number of 'conformal killing vectors', which are annihilated[4] by $P$. The conformal killing vectors correspond to reparametrisations which are actually Weyl scalings of the metric, which make our functional measure vanish.

To overcome this we define a reparametrisation to be expanded in only the non-zero eigenvectors of $P^\dagger P$, so that the integral over the metric becomes

$$\int \mathcal{D}g\frac{1}{\mathrm{Vol(Diff{\times}Weyl)}} = \int \mathcal{D}\rho{\prod_N}' \mathrm{d}b_N\sqrt{\mathrm{Det}\,'(U_N, P^\dagger P U_M)}\frac{1}{\mathrm{Vol(Diff{\times}Weyl)}}.$$

The prime indicates the exclusion of zero modes. Since we have taken care to make everything reparametrisation invariant, the integrand cannot depend on the choice of $U_N$ or $b_N$, the integral over which cancels the volume of the diffeomorphism group,

$$\int {\prod_N}' \mathrm{d}b_N = \frac{\mathrm{Vol(Diff)}}{\mathrm{Vol(CKV)}}. \tag{1.17}$$

[4]There is a distinction between proper conformal killing vectors and those vectors annihilated by $\nabla_a$, but this is somewhat technical and will not affect our results. For a discussion see [7].

This contains the volume of the group of conformal killing vectors since we excluded those vectors from our measure. We are left with only the integral over $\rho$ to perform,

$$\int \mathcal{D}g \frac{1}{\mathrm{Vol(Diff}\times\mathrm{Weyl)}} = \int \mathcal{D}\rho \frac{\sqrt{\mathrm{Det}'(P^\dagger P)}}{\mathrm{Vol(CKV)}} \frac{1}{\mathrm{Vol(Weyl)}}.$$

Following the classical theory we would hope that the integrand would be independent of the Liouville field, so that the remaining integral factor would cancel the volume of the group of Weyl transformations (much as with the $\delta\xi^a$ integral). However, as we have already pointed out the determinants do not appear to be Weyl invariant. To determine this dependence they must first be regulated, in a reparametrisation invariant manner.

To do this we generalise the finite dimensional result for $N \times N$ matrices $A$,

$$\log\det A = \mathrm{Tr}\,\log A \implies \delta\log\det A = \mathrm{Tr}\big(\delta A\, A^{-1}\big) = \int_0^\infty \mathrm{d}s\ \mathrm{Tr}\left(\delta A e^{-sA}\right), \quad (1.18)$$

which holds provided that the eigenvalues of the matrix are strictly positive. Under a Weyl scaling, the operators of interest to us transform as [8]

$$\begin{aligned} \delta\Delta &= -\delta\rho\Delta, \\ \delta P^\dagger &= -2P^\dagger\delta\rho, \\ \delta P &= P\delta\rho. \end{aligned}$$

The product of non-zero eigenvalues of these operators, denoted by a prime, are therefore

$$\begin{aligned} \delta\log\mathrm{Det}'\Delta &= -\int_\epsilon^\infty \mathrm{d}s\ \mathrm{Tr}\big(\delta\rho\Delta e^{-s\Delta}\big), \\ \delta\log\mathrm{Det}'P^\dagger P &= \int_\epsilon^\infty \mathrm{d}s\,\mathrm{Tr}\big[\,(-2P^\dagger\delta\rho P + P^\dagger P\delta\rho)e^{-sP^\dagger P}\big] \\ &= \int_\epsilon^\infty \mathrm{d}s\,\mathrm{Tr}\big[\,(-2\delta\rho PP^\dagger e^{-sPP^\dagger} + \delta\rho P^\dagger P)e^{-sP^\dagger P}\big]. \end{aligned}$$

We have used the cyclicity of the trace in the expression for $\mathrm{Det}'P^\dagger P$, and in each case we have introduced a small positive parameter $\epsilon$. This regulates the expression, since otherwise the lower end of the integration limit gives us

$$\mathrm{Tr}[\delta\rho] = \int \mathrm{d}^2\sigma\sqrt{g}\ \delta\rho \times \infty. \quad (1.19)$$

Performing the integrals gives, in the limit $\epsilon \to 0$,

$$\begin{aligned}\delta \log \mathrm{Det}'\Delta &= \mathrm{Tr}\big(\delta\rho e^{-\epsilon\Delta}\big),\\ \delta \log \mathrm{Det}'P^\dagger P &= -2\mathrm{Tr}\big[\delta\rho PP^\dagger e^{-\epsilon PP^\dagger}\big] + \mathrm{Tr}\big[\delta\rho P^\dagger P e^{-\epsilon P^\dagger P}\big].\end{aligned}$$

It is straightforward to see that, introducing the heat kernel $\mathcal{K}$ for the Laplacian,

$$\mathcal{K}(\sigma,\sigma';\epsilon) := \sum_n e_n^\mu(\sigma) e^{-\lambda_n\epsilon} e_{\mu n}(\sigma') \tag{1.20}$$

and such that $\Delta e_n = \lambda_n e_n$, we can write the regulated Laplacian determinant as

$$\delta \log \mathrm{Det}\,\Delta = -\int \mathrm{d}^2\sigma\sqrt{g}\;\delta\rho(\sigma)\mathcal{K}(\sigma,\sigma;\epsilon). \tag{1.21}$$

if there is no zero mode, and as

$$\begin{aligned}\delta \log \mathrm{Det}'\Delta &= -\int \mathrm{d}^2\sigma\sqrt{g}\;\delta\rho(\sigma){\sum_n}' e_n^\mu(\sigma)e^{-\lambda_n\epsilon}e_{\mu n}(\sigma'),\\ &= (e_0,\delta\rho e_0) - \int \mathrm{d}^2\sigma\sqrt{g}\;\delta\rho(\sigma)\mathcal{K}(\sigma,\sigma;\epsilon)\\ &= \delta\log\int \mathrm{d}^2\sigma\sqrt{g} - \int \mathrm{d}^2\sigma\sqrt{g}\;\delta\rho(\sigma)\mathcal{K}(\sigma,\sigma;\epsilon)\\ \Longrightarrow\ \delta\log\left(\frac{\mathrm{Det}'\Delta}{\int \mathrm{d}^2\sigma\sqrt{g}}\right) &= -\int \mathrm{d}^2\sigma\sqrt{g}\;\delta\rho(\sigma)\mathcal{K}(\sigma,\sigma;\epsilon)\end{aligned} \tag{1.22}$$

if there is, obtaining a regulated expression for the contribution in (1.12). The heat kernel obeys the relations

$$\Delta\mathcal{K} = -\frac{\partial}{\partial\epsilon}\mathcal{K}, \quad \lim_{\epsilon\to 0}\mathcal{K} = \frac{\delta^{(}2)(\sigma-\sigma')}{\sqrt{g}}, \tag{1.23}$$

which are independent of parameterisation, so our regulated determinants will be reparametrisation invariant, as we wished. Defining heat kernels for $P^\dagger P$ and $PP^\dagger$ allows us to regulate the metric contributions in a similar way.

The actual regularisation procedure, finding the small $\epsilon$ behaviour of the kernel to identify the divergence and finding the cut-off independent part to identify the regulated determinant, is well documented (see for example [9]). We will not go into the details here, since some examples of such calculations will feature in Chapter 3, but merely quote the result that

$$\delta\log\left[\frac{\sqrt{\mathrm{Det}'P^\dagger P}}{\mathrm{Vol(CKV)}}\left(\frac{\mathrm{Det}'\Delta}{\int \mathrm{d}^2\sigma\sqrt{g}}\right)^{-(D+1)/2}\right] = (D-25)\int \mathrm{d}^2\sigma\sqrt{g}\;\mathcal{R}\delta\rho(\sigma) \tag{1.24}$$

where $\mathcal{R}$ is the worldsheet Ricci scalar. It appears that we must now perform the integral over $\rho$, since the regulated integrals we have performed are not Weyl invariant. This poses a problem however, since the measure $\mathcal{D}\rho$ is given by

$$(\delta\rho, \delta\rho) = \int \mathrm{d}^2\sigma \sqrt{g}\, (\delta\rho)^2$$

which in the conformal gauge, for example, is

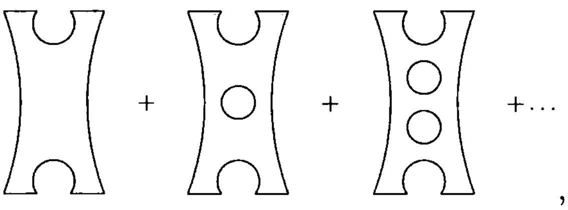
$$(\delta\rho, \delta\rho) = \int \mathrm{d}^2\sigma\, e^{\rho}(\delta\rho)^2.$$

No one understands how to interpret this measure, and so we cannot perform the final integral. It is not of a kind encountered in any quantum field theory. However, from (1.24) we see that if we are in $D+1 = 26$ spacetime dimensions the Weyl dependence disappears, and whatever the measure on $\rho$ the integral can only contribute and cancel the volume of the group of Weyl scalings.

Canonically, the critical dimension is that in which the Virasoro (physical state) conditions are sufficient to remove the unphysical negative norm states from the state space. In the functional approach, it is the dimension in which the metric contribution to the Weyl anomaly cancels that from the co-ordinates and allows us to complete the quantisation. Loop calculations are simpler in the functional approach since the metric integral automatically removes the negative norm state contributions.

Now that we know that when $D+1 = 26$ the functional integral has no Weyl dependence, we will usually regulate our determinants without worrying about the Weyl anomaly, using the zeta function regularisation technique.

To see how the mass spectrum arises in the functional approach it is necessary to study scattering amplitudes. The spectrum comprises those states which do not re-introduce the Weyl anomaly. Closed strings interact by splitting or joining at an interior point. Open strings interact by splitting at an interior point or joining at their endpoints. For example,



is the start of the sum over worldsheets for the four point function. The scattering amplitude for such processes is (1.7) on the appropriate worldsheets with the $W_i$ representing the interactions, as we will discuss. What makes computing these integrals feasible is the ability, only in the critical dimension, to deform the manifold into something simple. By a suitable choice of the Liouville mode we can deform one worldsheet into another. This is what accounts for the Veneziano amplitude property in string theory [10], namely that $t$–channel and $s$–channel scattering amplitudes are dual.

When the initial states come in from the infinite past and scatter to the infinite future, the Liouville mode can be chosen so that not only is the worldsheet deformed to a compact manifold, but the asymptotic strings are shrunk to punctures on this manifold's surface. At tree level the $N$–point tree level open string scattering amplitude is calculated by working on a disc with $N$ punctures on its boundary. The tree level closed string amplitude is a two sphere with $N$ punctures.

The states at the punctures are represented by local operators on the worldsheet, called vertex operators. These are the operators which excite a mode of the string. An open string can emit a closed string from any point on its worldsheet, and an open string from any point on the boundary of the worldsheet. The corresponding vertex operators are therefore integrated over the relevant parts of the worldsheet. These integrals must be made reparametrisation invariant. Examples are the open string tachyon vertex

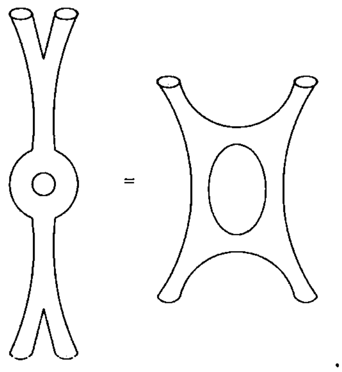
$$\int_{\text{boundary}} \mathrm{d}s\ e^{ik.X}.$$

and the open string photon vertex,

$$\int_{\text{boundary}} \mathrm{d}\sigma^a\ V(k)_\mu \partial_a X^\mu\, e^{ik.X}.$$

The former of these is not Weyl invariant, even as a classical expression, while the latter is. Terms outside the exponential can be represented as derivatives with respect to sources added to the string action. When we compute the Gaussian integral divergences arise because the classical part of the action contains the worldsheet Green's function which diverges at co-incident points. The regulation of these terms introduces a dependence on the Weyl anomaly, which only goes out if the vertex operator insertions obey the mass-shall conditions which are found in the canonical approach from the Virasoro constraints. In short, "the vertex operator plays the roll of an on-mass-shell interaction vertex" [10].

At tree level and one loop level, the symmetries of the string action allow us to deform any diagram into a state where each interaction vertex is coupled to an external line,



At two loops and beyond there will remain at least one vertex attached to three internal lines, which requires a description of off-shell strings.

Using the functional approach described forces us to deal with reparametrisation invariant string functionals to represent states. This is in general quite complex, and the Weyl anomaly can spoil our intuition of which states should be reparametrisation invariant[5].

Instead, the BRST quantisation techniques of Yang Mills theory can be generalised to string theory, where reparametrisation invariance is replaced by BRST invariance, which we now review. This will also afford us a good opportunity to introduce the complications of working on manifolds with more structure than the disk.

As mentioned earlier, string functional integrals are given by a sum over surfaces of increasing genus. In the critical dimension the integral is invariant under the gauge symmetries of reparametrisations and Weyl scalings. In general, however,

[5]For example, we shall explain in Chapter 5 how the Weyl anomaly is related to the puzzle of why only closed, and not open, strings appear to be independent of their parameterisation when they are contracted to points.

there exist deformations of the metric obeying $g^{ab}\delta g_{ab} = \nabla^a g_{ab} = 0$, orthogonal to reparametrisations and Weyl scalings[6]. For each solution there is a Teichmüller parameter, or moduli, which parameterises the transformation.

Consider the Faddeev-Popov approach to gauge fixing the functional integral over worldsheets with moduli. We may fix the gauge so that $g_{ab}$ is some reference metric, $\hat{g}_{ab}$, but this metric must depend on a set of moduli, call them $l_A$, so $\hat{g}_{ab} \equiv \hat{g}_{ab}(l_A)$. The number of moduli (the dimension of moduli space) depends on the properties of the worldsheet. For example, on a closed Riemann surface with $h > 1$ handles there are $6 - 6h$ moduli. We will make the Weyl invariant gauge choice $\sqrt{g}\, g^{ab} = \sqrt{\hat{g}}\, \hat{g}^{ab}(l_A)$ for some reference metric $\hat{g}(l_A)$. Ultimately, all string amplitudes should be independent of this choice.

An arbitrary variation of the metric can now be written

$$\begin{aligned} \delta g_{ab} &= \delta\rho\, g_{ab} + \nabla_{(a}\delta\xi_{b)} + \sum_A \delta l_A\, g_{ab,A}, \\ \Longrightarrow \delta(\sqrt{g} g^{ab}) &= -\sqrt{g} P(\delta\xi)^{ab} + \sum_A \sqrt{g}\, \delta l_A\, \chi_A^{ab} \end{aligned} \tag{1.25}$$

$g_{ab,A}$ is shorthand for $\partial g_{ab}/\partial l_A$ and $\chi_A^{ab}$ is the traceless symmetric part of $g^{ab}{}_{,A}$.

To fix the gauge we look for the Faddeev-Popov determinant which obeys

$$1 = \int \mathcal{D}g\ \Delta_{\mathrm{FP}}[g_{ab}]\ \delta\big[\sqrt{g}g^{ab} - \sqrt{\hat{g}}\hat{g}^{ab}(l_A)\big], \tag{1.26}$$

and the integral over $g$ is over the moduli and the group of diffeomorphisms and Weyl scalings,

$$\mathcal{D}g = \mathcal{D}\zeta \prod_A \mathrm{d}\, l_A \tag{1.27}$$

for a reparametrisation invariant measure $\mathcal{D}\zeta$ on the diff $\times$ Weyl group. Suppose that given $g_{ab}$ the constraints have a solution $\zeta = (\xi^a, \rho)$ and $\{l_A\}$. Expanding about this solution gives

$$1 = \int \mathcal{D}(\delta\rho, \delta\xi) \int \prod_A \mathrm{d}\delta l_A\ \Delta_{\mathrm{FP}}\ \delta\big[ - \sqrt{\hat{g}}\hat{P}(\delta\xi)^{ab} + \sum_A \sqrt{\hat{g}}\delta l_A \hat{\chi}_A^{ab}\big] \tag{1.28}$$

[6]There is an easily understood analogue for particle worldlines which is discussed in Section 2.2.

using equation (1.25). Now represent the delta functional by an integral over a (symmetric, traceless) covariant tensor $\lambda_{ab}$,

$$1 = \int \mathcal{D}(\lambda, \delta\rho, \delta\xi) \int \prod_A \mathrm{d}\delta l_A \; \Delta_{\mathrm{FP}} \; \exp\big(i \int \mathrm{d}^2\sigma \sqrt{\hat{g}} \; \lambda_{ab} \hat{P}(\delta\xi)^{ab} - \sum_A \delta l_A \lambda_{ab} \hat{\chi}^{ab}\big). \tag{1.29}$$

The integral over the Liouville mode goes out if we are in the critical dimension. We can invert the remaining expression by replacing bosonic with Grassmann variables to find

$$\Delta_{\mathrm{FP}}(\hat{g}) = \int \mathcal{D}(b_{ab}, c^a, \theta) \exp\big(i \int \mathrm{d}^2\sigma \sqrt{\hat{g}} \; b_{ab} \hat{P}(c)^{ab} - \sum_A \theta_A b_{ab} \hat{\chi}_A^{ab}\big). \tag{1.30}$$

The fields $\theta_A$, $c^a$ and $b_{ab}$ are a set of $A$ scalars, a vector and a traceless symmetric covariant tensor respectfully, all Grassmann valued. $b_{ab}$ will be referred to as the anti-ghost, for reasons which will shortly be apparent. If we integrate over the scalar fields we obtain

$$\Delta_{\mathrm{FP}}(\hat{g}) = \int \mathcal{D}(b_{ab}, c^a) \; \prod_A (\hat{\chi}_A^{ab}, b_{ab}) \; \exp\big(i \int \mathrm{d}^2\sigma \sqrt{\hat{g}} \; b_{ab} \hat{P}(c)^{ab}\big). \tag{1.31}$$

Now we expand the anti-ghost into zero modes and an orthogonal piece, $b = b^A \psi_A + b^\perp$, where the $b^A$ are Grassmann valued and the $\psi_A$ are the zero modes of $\hat{P}^\dagger$. The zero modes do not appear in the action (as they can be annihilated with integration by parts) but only in the leading product. Integrating out the zero modes leaves us with[7]

$$\Delta_{\mathrm{FP}}(\hat{g}) = \mathrm{Det}\,(\hat{\chi}_A, \psi^B) \int \mathcal{D}(b^\perp_{ab}, c^a) \exp\big(i \int \mathrm{d}^2\sigma \sqrt{\hat{g}} \; b^\perp_{ab} \hat{P}(c)^{ab}\big). \tag{1.32}$$

Now that we have identified the Faddeev Popov determinant, we insert the resolution of unity (1.26) into the functional integral for, say, the string partition function and obtain

$$\begin{aligned} \mathcal{Z} &\equiv \int \mathcal{D}(X, g) \frac{1}{\mathrm{Vol(Diff \times Weyl)}} e^{iS[X,g]} \\ &= \int \mathcal{D}(g, X) \frac{1}{\mathrm{Vol(Diff \times Weyl)}} \Delta_{FP}[g] \delta\big[\sqrt{g} g^{ab} - \sqrt{\hat{g}} \hat{g}^{ab}(l_A)\big] e^{iS[X,g]}. \end{aligned} \tag{1.33}$$

[7]This is easily verified by, for example, using the simplest case where there are two zero modes.

It is important to note that this applies to any functional we may choose, and we are only using the partition functional since it is the simplest example. Now, due to the invariance of the measures and the Faddeev Popov determinant under diff $\times$ Weyl transformations we can push the $g$ integration through the determinant and integrate it out,

$$\mathcal{Z} = \int \prod_A \mathrm{d}l_A \mathrm{Det}\,(\chi_A, \psi^B) \int \mathcal{D}(X, c, b^\perp)\ e^{iS[X,\hat{g}]+i\int\sqrt{\hat{g}}b_{ab}\hat{P}(c)^{ab}} \int \frac{\mathcal{D}\zeta}{\mathrm{Vol(Diff}\times\mathrm{Weyl)}}, \tag{1.34}$$

where the volume of the gauge group is explicitly cancelled, obtaining a well defined gauge fixed result.

We will now relate this to BRST quantisation. We take a step back to equation (1.30), and use this instead in the resolution of unity inserted into the partition function. We again represent the delta functional as a bosonic integral over $\lambda_{ab}$, and examine the expression

$$\mathcal{Z} = \int \prod_A \mathrm{d}l_A \int \mathcal{D}(X, \lambda, g, b, c, \theta) e^{iS_{FP}}, \tag{1.35}$$

where the Faddeev-Popov action is

$$\begin{aligned} S_{\mathrm{FP}} = S[X, g] + \int \mathrm{d}^2\sigma\sqrt{g}\ b_{ab}P(c)^{ab} + \int \mathrm{d}^s\sigma\lambda_{ab}\big(\sqrt{g}g^{ab} - \sqrt{\hat{g}}\hat{g}^{ab}(l_A)\big) \\ + \sum_A \theta_A \int \mathrm{d}^2\sigma b_{ab}\chi_A^{ab}. \end{aligned} \tag{1.36}$$

Defining an anticommuting operator $Q$ generating the transformations

$$\delta_Q X^\mu = c^a\partial_a X^\mu, \qquad \delta_Q\sqrt{g}g^{ab} = -\sqrt{g}P(c)^{ab}, \tag{1.37}$$

$$\delta_Q c^a = c^b\partial_b c^a, \tag{1.38}$$

$$\delta_Q\theta_A = 0, \tag{1.39}$$

$$\delta_Q b_{ab} = \lambda_{ab}, \qquad \delta_Q\lambda_{ab} = 0, \tag{1.40}$$

the Faddeev-Popov action can be written

$$S_{\mathrm{FP}} = S[X, g] + \big(\delta_Q - \theta_A\partial_A\big) \int \mathrm{d}^2\sigma\ b_{ab}\big(\sqrt{g}g^{ab} - \sqrt{\hat{g}}\hat{g}^{ab}(l_A)\big). \tag{1.41}$$

Up to the term in $\theta_A$ this action is precisely the BRST action we would write down if we wished to quantise the string action with a gauge fixing term forcing the metric to equal the reference metric.

The reference metric $\hat{g}$ is unaffected by the variation $\delta_Q$ and the derivative $\partial_A$. These transformations represent a reparametrisation with respect to the ghost $c^a$, and additional structure relating the anti-ghosts and constraints. The variation of $c^a$ is 'half' what one would expect for the transformation of a bosonic vector under a reparametrisation , and is required to maintain the nilpotency of the variation, $\delta_Q^2 = 0$.

The presence of the moduli means that the gauge fixed action is not BRST invariant, $\delta_Q S_{\rm FP} = \theta_A \partial_A S_{\rm FP}$. In fact this spoils the interpretation of physical states as living in the cohomology of $Q$, see [11]. The symmetry is still useful – it encodes the reparametrisation invariance as BRST invariance which can still be used to quantise the theory.

## 1.5 String field theory

As we mentioned, string theory is analogous to quantum mechanics, in that it is first quantised. Even though the theory generates its own loop expansion in the sum over topologies, this is still a perturbation series. Consider the case of Yang-Mills theory, where the perturbative expansion shows no sign of confinement. That string theory contains such non-perturbative information is certain, as illustrated by the web of old and new [12] string dualities. To get at such information we need a non-perturbative formulation of strings, a string field theory.

Another example of why a string field theory is needed also illustrates the major problem with constructing such theories. Suppose that rather than the flat target spacetime of the Polyakov action given earlier we wish to include a spacetime metric $G_{\mu\nu}$. The action we would use is an example of a non-linear sigma model,

$$\int \mathrm{d}^2\sigma\sqrt{g}\ \partial_a X^\mu g^{ab}\partial_b \mathbf{X}^\nu G_{\mu\nu}(X).$$

This metric is put in by hand, however, and the Euler Lagrange equations impose no restrictions on it – so where do Einstein's equations come from? As with the dimension of spacetime, requiring Weyl invariance at the quantum level imposes restrictions on $G_{\mu\nu}$ which at leading order are Einstein's equations. The full constraint on $G_{\mu\nu}$ receives contributions from all orders in perturbation theory, and so is not yet known. The constraint, and those for other fields we may include in the model such as the Dilaton, would be given us to as equations of motions by the string field theory.

Herein lies the problem; we would look for a string field theory which reproduced the perturbation expansion of our conformal field theory. The string field would be a weighted sum over states of the conformal field theory, i.e. we require a conformal field theory before we can construct a string field theory. This is called the problem of background dependence. What we are doing in constructing such theories is expanding the true string field action around some classical solution and analyzing the resultant action for fluctuations (our string field) about this solution. As yet no-one knows what the true action is, nor if we might somehow reconstruct it from the string field actions we have. We do not attempt to tackle such problems here, we will work with string field theories which are based on the conformal background of the flat space Polyakov action.

We briefly mention the problems of closed string field theory(see [13] and the references therein for a fuller discussion). The problem of constructing interacting closed string field theories was originally the definition of an interaction vertex which preserved reparametrisation invariance while leaving the action quantisable. If we were to attempt a Witten type construction, we would need to choose a point on the closed string to be the 'midpoint' for the local interaction to occur, but this would break manifest reparametrisation invariance.

Once it was understood how the closed strings interacted the problem then became that of showing that the Feynman rules of the string field theory correctly reproduced the diagrams of string theory – i.e. that the rules gave a decomposition of the space of all Riemann surfaces [14]. The theory is somewhat more complicated than the open string theories we will discuss, and are more geometrical in nature (that geometry should be so important can be regarded as natural since it is in the closed string spectrum that the graviton appears). The quantisation of closed string field theory seems to require the full weight of the Batalin-Vilkovisky formalism [15]. We will not elaborate on the details, since our methods will apply to both open and closed strings equally.

Let us turn to the details of some open string field theories. The string field is a functional $\phi[X]$ of the string $X^{\mu}(\sigma)$, independent of the co-ordinate $\sigma$ used to parameterise the string. Suppose we had some Lagrangian density $\mathcal{L}$ for the string field. To quantise we would identify the conjugate momentum $\pi[X]$ and impose the canonical commutation relations $[\phi, \pi] = i\delta$ at equal time. There is a problem with this since the string field is defined not on spacetime points but on spacetime curves. The conjugate momentum is defined as

$$\pi[X, \sigma] = \delta\mathcal{L}/\delta\big(\partial\phi[X]/\partial X^0(\sigma)\big) \tag{1.42}$$

since $X^0$ is time – but at what value of $\sigma$ do we take the time derivative? There are an infinite number of choices. Not only does this make the conjugate momentum badly defined but brings in an explicit dependence on the parameterisation of the string. In short, there is no well defined time for the string field, so we cannot canonically quantise the theory.

One way out of this problem is to give up manifest Lorentz covariance. In fixing the conformal gauge in string theory there is actually a residual gauge symmetry. If we choose a Weyl scaling $\rho$ and a reparametrisation $\xi^a$ such that

$$\partial^{(a}\xi^{b)} = \rho\hat{g}\eta^{ab}$$

then the conformal gauge is preserved. The light-cone gauge uses up this residual symmetry to identify a unique time for the string. On the worldsheet the residual symmetry corresponds to changes in the co-ordinates $\sigma^+ \to \tilde{\sigma}^+(\sigma^+)$, $\sigma^- \to \tilde{\sigma}^-(\sigma^-)$ where $\sigma^{\pm} = \sigma \pm \tau$. This implies that we can perform a gauge transformation which makes $\tilde{\tau}$ a solution of the free wave equation, and means we can gauge fix so that worldsheet $\tau$ is equal to one of the co-ordinates of the string.

In the light-cone quantisation we define

$$X^{\pm} \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^{25}), \tag{1.43}$$

and identify $X^+ = \tau$ as the physical time. The remaining degrees of freedom are the 24 transverse co-ordinates $\mathbf{X}$, while $X^-$ is determined by the Virasoro constraints up to its centre of mass piece. Taking $\tau$ as the physical time direction allows us to define an unambiguous momentum,

$$\pi[X] = \delta\mathcal{L}/\delta\big(\partial\phi[X]/\partial\tau\big). \tag{1.44}$$

It is customary to work in a mixed co-ordinate/ momentum representation in which we exchange the centre of mass of $X^-$ with its Fourier momentum $p_+$, denoting such a string field $\phi_{p+}[\mathbf{X},\tau]$.

The light-cone gauge also helps us construct the string field Lagrangian by highlighting some of the properties it should have. For example, since we have identified worldsheet time with physical time we would expect the generator of first quantised time evolution to generate time translations of the second quantised field also,

$$i\frac{\partial}{\partial\tau}\phi_{p+}[\mathbf{X},\tau] = \frac{(L_0-1)}{2p_+}\phi_{p+}[\mathbf{X},\tau] \tag{1.45}$$

where $L_0$ is the normal ordered zeroth Virasoro generator and the factor of $2p_+$ is a convention. The free string field action which reproduces this is [16]

$$S = \int \mathrm{d}\tau \int_0^\infty \mathrm{d}p_+ \int \mathcal{D}\mathbf{X}^{(24)}\ i\phi^\dagger_{p+}\partial_\tau\phi_{p+} - \frac{1}{2p_+}\phi^\dagger_{p+}h\phi_{p+} + S_{\mathrm{int}}[\phi^\dagger_{p+},\phi_{p+}] \tag{1.46}$$

where $h$ is the first quantised string Hamiltonian,

$$h = \frac{1}{2}\int_0^\pi \mathrm{d}\sigma\ \mathbf{X}'(\sigma)^2 - \frac{\delta^2}{\delta\mathbf{X}(\sigma)^2}. \tag{1.47}$$

The conjugate momentum is $i\phi^\dagger_{p+}$ and canonical commutation relations are

$$\big[\phi_{p+}[\mathbf{X}_1,\tau],\phi^\dagger_{q+}[\mathbf{X}_2,\tau]\big] = \delta(p_+-q_+)\prod_\sigma \delta\big[\mathbf{X}_1(\sigma)-\mathbf{X}_2(\sigma)\big]. \tag{1.48}$$

To solve the equations of motion the transverse co-ordinates and therefore the Schrödinger equation can be expanded in Fourier modes,

$$\mathbf{X} = \frac{1}{\sqrt{2\pi}}\bigg(\mathbf{x}_0 + 2\sum_{n=1}^\infty \mathbf{x}_n\cos(n\sigma)\bigg)$$

in which case the equation for each mode $x^i_n$ is that of a harmonic oscillator of frequency $n$. The string field then has the mode expansion

$$\phi_{p+} = \int\frac{\mathrm{d}^{24}\mathbf{p}}{(2\pi)^{24}}\sum_{\{l_{n,i}\}} A_{\mathbf{p},p_+,\{l_{n,i}\}}e^{-iE_-\tau}e^{i\mathbf{p}.\mathbf{x}_0}\prod_{n,i}\frac{H_{l_{n,i}}(\sqrt{n}x^i_n)}{(2^l l!)^{1/2}(\pi/n)^{1/4}}e^{-n(x^i_n)^2/2}. \tag{1.49}$$

The product over $n$ and $i$ is the product over the separated Hermite solutions for each string mode $x_n$ in dimension $i$. There is then a sum over all possible combinations of assigning excitation level values $l$ to each of these solutions, representing the infinite number of independent solutions to the full equations of motion.

Despite the complexity of the functions in the above this is in the form "sum over creation/ annihilation operators multiplied by wave functions" as found in quantum field theory. In first quantised string theory the creation operator $\alpha_n^\dagger$ excited an oscillation of the string, here the operator $A^\dagger$ creates an entire string from the vacuum.

It can be shown [16] that second quantised Poincaré generators obeying the correct commutation relations and consistent with the light cone gauge choice can be constructed in the theory. This ensures that despite the non-covariant gauge choice Lorentz covariance is preserved and completes the quantisation of the free string field.

The free propagator for the string field between times $\tau_2$ and $\tau_1$ can be calculated from the first quantised Polyakov integral on a strip of length $\tau_2 - \tau_1$ with delta functional sources at each end representing prescribed initial and final strings (we will return to this in the covariant theory in Chapter 3).

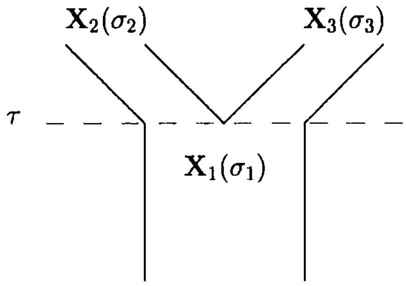
This suggests that the interaction for the string field theory should represent the splitting and joining of such strips. The action which allows strings to interact by splitting or joining at an interior point is

$$\begin{aligned}\frac{\beta}{2}\int \mathrm{d}\tau \int \prod_{j=1}^{3} \mathcal{D}\mathbf{X}_j \mathrm{d}p_{+j}\delta(p_{+3} - p_{+2} - p_{+1})\phi^\dagger_{p_{+3}}[\mathbf{X}_3,\tau]\phi^\dagger_{p_{+2}}[\mathbf{X}_2,\tau]\phi_{p_{+1}}[\mathbf{X}_1,\tau] \\ \times \prod_{\sigma_1} \delta\big(\mathbf{X}_1(\sigma_1) - \mathbf{X}_2(\sigma_2)\theta(\sigma_1 - \pi/2) - \mathbf{X}_3(\sigma_3)\theta(\pi/2 - \sigma_1)\big) + \text{h.c.}\end{aligned} \tag{1.50}$$

where $\theta$ is the Heaviside function. The strings obey the parameterisation constraints

$$\begin{aligned}\sigma_2 &= 2\sigma_1 \qquad 0 < \sigma_1 < \pi/2, \\ \sigma_3 &= 2\sigma_1 - \pi \qquad \pi/2 < \sigma_1 < \pi.\end{aligned}$$

The delta functional interaction conserves $p_+$ and forces the arguments $\mathbf{X}_j$ to overlap as shown below.




This interaction is local in our chosen time, as illustrated. Although this appears to destroy the smoothness of string interactions, it is equivalent to inserting a Y-shaped worldsheet like that shown above between three string propagators and taking a limit as the thickness of this Y region vanishes. For open strings it is necessary to also introduce a quartic interaction, for details see [16]. In general locality in a single direction does not re-introduce the ultraviolet divergences of particle theory which strings are supposed to smooth away.

The locality of string field theory in the light-cone gauge is viewed with some caution in the literature. It is unconvincing that gauge fixing alone could remove non-locality from the theory. It has been suggested [17] that this locality, and also the agreement of light-cone string field theory with on-shell string theory amplitudes is an artifact of perturbation theory only. There is no evidence that the light-cone string field theory contains any non-perturbative information. Lorentz invariance of the theory is also expected to only hold perturbatively.

One of the most successful covariant approaches to string field theory was given by Witten [18]. He proposed the open string field action

$$S = \frac{1}{2}\int \Phi \star Q\Phi + \frac{\beta}{3}\int \Phi \star \Phi \star \Phi. \tag{1.51}$$

The string field is $\Phi$, $Q$ is the open string BRST operator, $\beta$ is the string coupling and $\star$ is an associative, graded product on the space of string fields. The integration maps from the space of string fields to the complex numbers, and $g$ is the string coupling constant. The free field equations of motion are just

$$Q\Phi = 0, \tag{1.52}$$

and there is a gauge invariance of the action under

$$\delta\Phi = Q\Lambda \tag{1.53}$$

for any string field $\Lambda$. This tells us that the free string field is BRST invariant and has a gauge symmetry under reparametrisations of the string. Thus the free string field reproduces the theory of the free string. The free string field, therefore, can be expanded as a sum over fields associated to open string Fock space states,

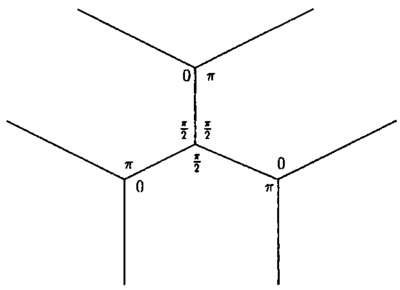
$$|\,\Phi\,\rangle = \int \frac{\mathrm{d}^{26}p}{(2\pi)^{26}}\,\phi(p)|\,p\,\rangle + A_\mu(p)\alpha^\mu_{-1}|\,p\,\rangle + \ldots \tag{1.54}$$

where $\phi(p)$ is the tachyon field, $A^\mu(p)$ is the gauge field, etc. The ghost and antighost states must also be included in this expansion.

The star product combines a pair of string functionals by sewing the left half of one string to the right half of another (the open strings have parameter range $[0,\pi]$),

$$\begin{aligned}\Phi\star\Theta[Z(\sigma)] = \int \prod_{0\le\sigma'\le\pi/2} &\mathcal{D}Y(\sigma')\mathcal{D}X(\pi-\sigma')\\ \times \prod_{\pi/2\le\sigma''\le\pi} &\delta\big[X(\sigma'')-Y(\pi-\sigma'')\big]\,\Phi[X(\sigma'')]\Theta[Y(\sigma'')]\end{aligned}$$

where $Z(\sigma) = X(\sigma)$ for $0\le\sigma\le\pi/2$ and $Z(\sigma)=Y(\sigma)$ for $\pi/2\le\sigma\le\pi$. The interaction vertex joins three strings at their midpoints, most easily illustrated by




It is possible to write the string field action explicitly as an infinite series. The free part is an infinite sum over the field theory actions for each field in the expansion above. The interaction term become a sum over interactions between this infinite set of fields. The first few terms of the action are

$$\begin{aligned}&\frac{1}{2}\int \mathrm{d}^{26}x\;\partial\phi(x)\cdot\partial\phi(x) + \phi^2(x) + \partial A^\mu(x)\cdot\partial A_\mu(x)\\ &\qquad + \frac{\beta}{3}\int \mathrm{d}^{26}x\;\left(e^{a\partial^2}\phi(x)\right)^3 + \left(e^{a\partial^2}\phi(x)\right)\left(e^{a\partial^2}A^\mu(x)\right)\left(e^{a\partial^2}A_\mu(x)\right)\end{aligned} \tag{1.55}$$

where $a = \log(27/16)/2$ (for the origins of this number see Sections 6.4, 6.5 of [19]). These terms describe the tachyon with negative mass squared, the photon, with zero mass, and the cubic interactions between them. In terms of these component fields the interaction term implies that string field theory behaves very differently to any ordinary quantum field theory. There appear to be an infinite number of non-renormalisable interactions, which somehow combine into the reasonable interpretation of strings joining as described above. It is possible to show that Witten's theory correctly reproduces all open string amplitudes, including the measures on moduli space [20], [21], [22].

We now turn to quantising the theory. In the free theory the centre of mass of $X^0$ is taken as time and can be used to quantise without complication [23], since the free action looks like $\Phi p_0^2 \Phi$.

When we introduce the interaction vertex we run into problems, since the interaction ties together three strings at their midpoint, which in general differs from the centre of mass position. The interaction is non-local in our chosen time, and the interaction vertex contains infinite orders of time derivatives, which prevents us quantising canonically (see [17] for problems with higher derivative theories)

It is logical to instead treat the midpoint of the string as the time co-ordinate. The interaction vertex is local with respect to $X^0(\pi/2)$ and it appears our problems are solved. Unfortunately, when we write the free piece of the action in terms of the midpoint and the remaining modes, the canonical momentum picks up a divergence proportional to the one dimensional delta function $\delta(0)$. To proceed, the divergence must be regulated [24] which can be achieved by discretising the string.

It is possible to quantise the theory and avoid these divergences [24], [25]. To quantise we would like to have a vertex which is local in time, but the spatial part of the vertex can remain non-local and not affect the quantisation procedure. To this end, the light-cone components $X^0 \pm X^{25}$ are re-written in midpoint form, and the remaining co-ordinates are left in the standard 'centre of mass' basis. In this representation the vertex is local in $X^+(\pi/2)$ and the kinetic term has no divergence.

It seems at any rate that non-locality is a feature of string field theory which cannot be argued away without introducing divergences and other problems. The quantum field theories with which we deal in this thesis are, for the most part, local. However, in later sections we explain how to generalise our ideas to non-local theories, and then to string field theory.

Briefly, let us summarise some more modern areas of investigation in string field theory. Much of the recent work has focussed on the Sen conjectures. The tachyon in the open string spectrum indicates, as with quantum field theory, that we are trying to perform perturbation theory around an unstable point of the potential which governs the full, non-perturbative string theory. Sen observed that the open string can be thought of as ending on a space filling D25-brane, and that since this Brane carries no conserved charges it is unstable. He conjectured that the tachyon was the unstable mode of the D25-brane [26]. More precisely, Sen's conjectures are that the stable minimum of the tachyon potential has an energy density equal to minus the tension of the D25-brane, as measured relative to the unstable vacuum. That is, in the true vacuum the tachyon condenses and annihilates the D25 brane, leaving a closed string vacuum in which there are no open string states.

It was suggested that Witten's cubic field theory could be used to test these conjectures by calculating the energy density of the vacuum and showing that it gave the tension of the brane. Investigations of this result have been encouraging, but due to the complexity of Witten's theory the result has not yet been proven analytically.

There are encouraging numerical results however. Defining he level of a state in the string field expansion to be its eigenvalue under $L_0$ relative to the tachyon state, the (N,M) level truncation of the action neglects all fields of level greater than $N$ and all interactions between fields whose combined level is greater than $M$. This leaves us with a finite number of terms in the action which can be analysed numerically. The complexity of the calculations grows exponentially with $N$, the numerical results seem to converge rapidly to Sen's projected value. Even the most naive calculation, dropping everything except the tachyon and restricting the field to a static configuration gives a vacuum energy of around 70% of Sen's value. For recent results of these calculations see [27] [28] [29] and for a review see [30]. For a general review of calculational methods in Witten's SFT see [31] and references therein.

For supersymmetric generalisations of Witten's theories, modifications to the star product and integral are necessary to avoid problems with ghost number counting [32], but these modifications then lead to a breakdown of gauge invariance, see for example [33]. The problem can be avoided by working in a larger ghost space [34]. The string field action then has a Wess Zumino Witten type structure [35]. There has recently been an interest in work connected to this approach and directed at the heterotic string, see [36] and [37]. Again, Batalin Vilkovisky techniques are required and the theory is quite complex. At leading order the non-linear equations of motion and gauge invariance of this string field theory reproduce those of the heterotic string, and it is hoped the theory can be used for investigating the closed string tachyon. The construction of superstring field theories remains, in general, very much open for investigation. For example, there is no known string field action which reproduces the amplitudes of the type II superstring.

# Chapter 2

# Free quantum field theory

In this chapter we give a graphical description of time dependence in free field theories which we will later generalise to the string field. Our approach is based on interpreting sums over field histories in terms of sums over particle histories as we will describe. We begin with some definitions, then prove the 'gluing property' which will be of paramount importance in the remainder of this chapter where we present various calculations to demonstrate that it determines time dependence in the free field theory.

## 2.1 The Schrödinger functional and the vacuum

The Schrödinger functional is defined as

$$\begin{aligned} \mathcal{S}[\phi_2, t_2; \phi_1, t_1] &= \langle\, \phi_2\, | e^{-i\hat{H}(t_2-t_1)/\hbar} |\, \phi_1 \,\rangle \\ &= \langle\, D\, | e^{i\int \phi_2 \hat{\pi}/\hbar} e^{-i\hat{H}t/\hbar} e^{-i\int \phi_1 \hat{\pi}/\hbar} |\, D \,\rangle \end{aligned} \tag{2.1}$$

where the states $\langle\, \phi_i \,|$ are eigenvectors of the field operator. This functional evolves arbitrary functionals through time as follows. Suppose we have some state $\Psi[\phi, t_1]$. Then this state at later time $t_2$ is, inserting a complete set,

$$\begin{aligned} \Psi[\phi, t_2] &\equiv \langle\, \phi\, | e^{-i\hat{H}(t_2-t_1)/\hbar} |\, \Psi \,\rangle \\ &= \langle\, \phi\, | e^{-i\hat{H}(t_2-t_1)/\hbar} \left[ \int \mathcal{D}\varphi \,|\, \varphi \,\rangle\langle\, \varphi \,| \right] |\, \Psi \,\rangle \\ &= \int \mathcal{D}\varphi \,\langle\, \phi\, | e^{-i\hat{H}(t_2-t_1)/\hbar} |\, \varphi \,\rangle\langle\, \varphi \,|\, \Psi \rangle \equiv \int \mathcal{D}\varphi\; \mathcal{S}[\phi, t_2; \varphi, t_1]\, \Psi[\varphi, t_1]. \end{aligned} \tag{2.2}$$

From (2.1) the Feynman prescription for the Schrödinger functional is

$$\mathcal{S}[\phi_2, t_2; \phi_1, t_1] = \int \mathcal{D}\varphi \; e^{iS[\varphi]/\hbar} \Big|_{\varphi(\mathbf{x},t_1)=\phi_1(\mathbf{x}).}^{\varphi(\mathbf{x},t_2)=\phi_2(\mathbf{x})} \tag{2.3}$$

The change of variable

$$\tilde{\varphi} := \varphi + \theta(t - t_2)\phi_2(\mathbf{x}) + \theta(-t + t_1)\phi_1(\mathbf{x}) \tag{2.4}$$

where $\theta$ is the step function ($\theta(0) = 1$), leaves the measure invariant and will move the $\phi$–dependence in the integration limits to boundary terms in the action. Naively the $\theta$ terms do not contribute to the potential, since the action is evaluated between times $t_1$ and $t_2$. Our functional integral becomes, dropping the tilde,

$$\begin{aligned}\int \mathcal{D}\varphi \, \exp\bigg(\frac{i}{\hbar}S[\varphi] + \frac{i}{\hbar}\int \mathrm{d}^D\mathbf{x}\,\phi_2(\mathbf{x})\,\dot{\varphi}(\mathbf{x},t_2) - \frac{i}{\hbar}\int \mathrm{d}^D\mathbf{x}\,\phi_1(\mathbf{x})\,\dot{\varphi}(\mathbf{x},t_1) \\ + \frac{i}{2\hbar}\int \mathrm{d}^D\mathbf{x}\,\phi_2(\mathbf{x})^2\delta(0) - \frac{i}{2\hbar}\int \mathrm{d}^D\mathbf{x}\,\phi_1(\mathbf{x})^2\delta(0)\bigg)\end{aligned} \tag{2.5}$$

and the integration variable now obeys $\varphi = 0$ on the boundaries $t = t_1$ and $t = t_2$. There are several methods of obtaining this result, such as shifting the field by a classical solution or by smoothed step functions, see [38] for a full discussion.

The first three terms in (2.5) relate directly to the canonical expression in (2.1). The final two terms require regularisation. As Symanzik has discussed [39], placing source terms on the boundary leads to divergences, and the Schrödinger functional is an example. The divergences appear in perturbation theory because the field is placed at the same point as the image charges which enforce boundary conditions, and are in addition to the usual free space UV divergences. In order to regulate the image divergences we should split in time the fields– thus the fields in (2.5) should be defined at different ordered times, with the difference acting as regulator.

We will take this time splitting regularisation to be understood so that we may drop the delta functions in (2.5) and standard field theory results imply that the logarithm of the Schrödinger functional is given by the sum of connected diagrams built from a propagator $G_D$ which vanishes on the initial and final time hypersurfaces, where all external legs are constrained to end, and interaction vertices are integrated only over the interval $[t_1, t_2]$. We will discuss interactions in a later chapter, for now let us carry out the free field integral in (2.5) and introduce some notation.

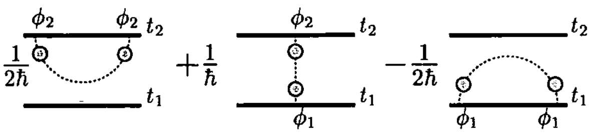
Using the usual $i\varepsilon$ prescription to make the operator in the exponent positive definite the Gaussian converges to

$$\begin{aligned} \mathbb{S}[\phi_2, t_2; \phi_1, t_1] = N_{\mathbb{S}} \exp\Big( -\frac{1}{\hbar} \iint \mathrm{d}^D(\mathbf{x}, \mathbf{y})\ & \frac{1}{2}\phi_2(\mathbf{y}) G_D(\mathbf{y}, \dot{t}_2, \mathbf{x}, \dot{t}_2)\phi_2(\mathbf{x}) \\ & -\phi_2(\mathbf{y}) G_D(\mathbf{y}, \dot{t}_2, \mathbf{x}, \dot{t}_1)\phi_1(\mathbf{x}) \\ & +\frac{1}{2}\phi_1(\mathbf{y}) G_D(\mathbf{y}, \dot{t}_1, \mathbf{x}, \dot{t}_1)\phi_1(\mathbf{x}) \Big) \end{aligned} \tag{2.6}$$

where the propagator $G_D$ obeys Dirichlet boundary conditions on the surfaces $t = t_1$ and $t = t_2$ as described above, reflecting the fact that $\langle\, D\,|\hat{\phi}(\mathbf{x}) = 0$. $G_D$ appears differentiated since $\phi$ is coupled to $\dot{\varphi}$ in the integral,

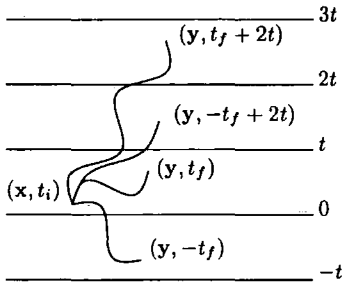
$$G_D(\mathbf{x}_2, \dot{t}_2, \mathbf{x}_1, \dot{t}_1) \equiv \frac{\partial^2}{\partial t\, \partial t'} G_D(\mathbf{x}_2, t, \mathbf{x}_1, t')\Big|_{t=t_2,\ t'=t_1} . \tag{2.7}$$

The normalisation constant $N_{\mathbb{S}}$ will be discussed below. The Feynman diagram expansion of the free field Schrödinger functional can be written

$\mathbb{S}[\phi_2, t_2; \phi_1, t_1] = N_{\mathbb{S}} \exp\Big( -\frac{1}{2\hbar}$  $\Big)$ . (2.8)

The heavy lines denote the boundaries at $t = t_1, t_2$. In our diagrams a dotted line will denote a propagator with Dirichlet conditions on all boundaries shown (whether or not the propagator ends on them all). The grey dot is the time derivative.

The Schrödinger functional between times 0 and $t$ (without loss of generality) is built from the Green's function $G_D$ which vanishes on the hypersurfaces at times 0 and $t$. This can be interpreted in terms of first quantisation as follows. Beginning in free space we identify points with their images under an $\mathbb{S}^1/\mathbb{Z}_2$ (orbifold) compactification of the time direction, radius $t/\pi$. In a sum over paths from points $(\mathbf{x}, t_i)$ to $(\mathbf{y}, t_f)$ on this spacetime we must include the paths to the image points since they are considered equivalent. If we attach a minus sign each time a path crosses a reflection of the quantisation surfaces at times $nt$ for $n \in \mathbb{Z}$,




then the sum over paths to each image gives the free space propagator, call it $G_0$, weighted with a sign. The sign is positive if the image is of form $t_f + 2nt$ and negative if of form $-t_f + 2nt$ for $n \in \mathbb{Z}$. All together, the sum over paths gives the sum

$$\sum_{n\in\mathbb{Z}} G_0(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2 + 2nt) - G_0(\mathbf{x}_1, t_1; \mathbf{x}_2, -t_2 + 2nt) \equiv G_D(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2) \qquad (2.9)$$

which is equal to the desired propagator $G_D$ because the sum on the left hand side is the method of images imposition of the boundary conditions implicit on the right hand side. Since the sum over paths generalises to the Polyakov sum over worldsheets in string theory, we suggest that the string field Schrödinger functional has a similar description.

The method of images implies that the Schrödinger functional can be interpreted in terms of sums of ordinary free space Feynman diagrams. Later in this chapter we will use this interpretation to show that time evolution can be described using properties of first quantisation. It is this structure which we will later generalise to string field theory. To do this, we will also need the vacuum state wave functional. The conventional method of constructing the vacuum wave functional is via a large time path integral [39]. If we apply the time evolution operator $\exp(-i\hat{H}t/\hbar)$ to any state $|\, v\,\rangle$ not orthogonal to the vacuum, then for large times

$$\exp(-i\hat{H}t/\hbar)|\, v\,\rangle \sim |\, 0\,\rangle e^{-iE_0 t/\hbar}\langle\, 0\,|\, v\,\rangle \qquad (2.10)$$

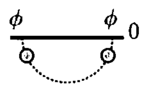
where $E_0$ is the energy of the vacuum[1]. Thus

$$\Psi[\phi] = \lim_{t \to \infty} N e^{iE_0 t/\hbar} \langle D \,|e^{i\int dx\, \phi(\mathbf{x})\hat{\pi}(\mathbf{x},0)/\hbar} e^{-i\hat{H}t/\hbar} |\, v \rangle. \tag{2.11}$$

We choose the normalisation constant $N$ to remove dependence on the choice of $v$, so $N = \langle 0 \,|\, v \rangle^{-1}$. We obtain the functional integral representation following arguments similar to those above,

$$\Psi_0[\phi] = \int \mathcal{D}\varphi(\mathbf{x},t) \exp\left( \frac{i}{\hbar} S[\varphi] + \frac{i}{\hbar} \int d^D\mathbf{x}\, \dot{\varphi}(\mathbf{x},0)\phi(\mathbf{x}) \right)\Bigg|^{\varphi(\mathbf{x},0)=0} \tag{2.12}$$

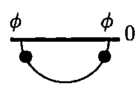
The boundary condition on the field at $t = -\infty$ is that it must be regular. The overall normalisation of the vacuum will not concern us here, so we set it equal to one. The free field Gaussian integral gives

$$\Psi_0^{\text{free}}[\phi] = \exp\Bigg( - \frac{1}{2\hbar}$$  $$\Bigg) \tag{2.13}$$

where the propagator, call it $G_{d(0)}$, satisfies Dirichlet conditions on the boundary $x^0 = 0$. By the method of images this propagator is

$$G_{d(0)}(\mathbf{x}, t_1, \mathbf{y}, t_2) = G_0(\mathbf{x}, t_1, \mathbf{y}, t_2) - G_0(\mathbf{x}, t_1, \mathbf{y}, -t_2) \tag{2.14}$$

The differentiation leads to each term contributing equally when the propagator ends on the boundary and so we can write the free vacuum wave functional in terms of the free space propagator. Let an unbroken line indicate the free space propagator and a black dot denote the time derivative with a factor of $-2$ as appears in the gluing properties. Then the free vacuum functional is

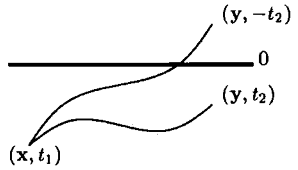
$$\Psi_0^{\text{free}}[\phi] = \exp\Bigg( - \frac{1}{4\hbar}$$  $$\Bigg). \tag{2.15}$$

Using this we can verify the time independence of the vacuum wave functional; for propagators $G_{d(t)}$ which obey Dirichlet conditions on the plane at time $t$ we have

$$\ddot{G}_{d(t_1)}(\mathbf{x}, t_1, \mathbf{y}, t_1) = \ddot{G}_{d(t_2)}(\mathbf{x}, t_2, \mathbf{y}, t_2). \tag{2.16}$$

[1]Strictly, we should make $i\hat{H}$ positive definite by the addition of some regulator and later let it tend to zero to achieve this result. This is what happens in the Euclideanised theory which we will consider later.

To complete the discussion of the vacuum wave functional we describe how the propagator is given in first quantisation. If we identify spacetime points with their reflection in $t = 0$, the quantisation surface, then when we sum over paths from $(\mathbf{x}, t_1)$ to $(\mathbf{y}, t_2)$ we have to include paths from $(\mathbf{x}, t_1)$ to $(\mathbf{y}, -t_2)$, the reflection of $(\mathbf{y}, t_2)$,




Now weight paths with a minus sign each time they cross the surface $t = 0$. Paths from $(\mathbf{x}, t_1)$ to $(\mathbf{y}, -t_2)$ must cross the quantisation surface an odd number of times and acquire an overall minus sign, whereas paths directly from $(\mathbf{x}, t_1)$ to $(\mathbf{y}, t_2)$ cross an even number of times and are weighted with an overall plus sign. The contribution from the latter paths gives $G_0(t_1, t_2)$ and from the former give $-G_0(t_1, -t_2)$ in expression (2.14) for the Dirichlet propagator $G_{d(0)}$.

Our goal is to construct the Schrödinger functional and vacuum functional for string field theory – of course we do not necessarily know the Hamiltonian so the above definition does not seem immediately applicable. However, we will find that the Feynman diagram expansions we have given must hold for string field theory also, if we generalise a key property of propagators which we now discuss.

## 2.2 The gluing property

It is well known that the off-shell free space propagator from the point $x_i \equiv (\mathbf{x}_i, t_i)$ to $x_f \equiv (\mathbf{x}_f, t_f))$ may be written as a sum over all paths from $x_i$ to $x_f$ with an action involving an intrinsic metric $g$ [6] [2]. Integrating out $g$ gives a Boltzmann weight equal to the exponential of the length of the path,

$$\begin{aligned} G_0(x_f; x_i) &= \int \frac{\mathcal{D}(x, g)}{\mathrm{Vol\ Diff}}\, e^{i\int_0^1 d\xi \sqrt{g}\,(\dot{x}\cdot\dot{x}/(2g)+m^2/2)}\Bigg|_{x(0)=x_i}^{x(1)=x_f} \\ &= \int \mathcal{D}x\; e^{im\int_0^1 d\xi\, \sqrt{\dot{x}\cdot\dot{x}}}\Bigg|_{x(0)=x_i}^{x(1)=x_f} . \end{aligned} \tag{2.17}$$

The path is parameterised by $\xi$ and $x(0)$ is the point $x_i$, $x(1)$ the point $x_f$. We have corrected for the over counting of equivalent paths resulting from reparametrisation invariance of the action by dividing out the volume of the space of reparametrisations $f(\tau)$. This is the particle analogue of the Polyakov integral in string theory. Any metric can be reduced, by a suitable gauge transformation, to

$$g(\xi) = T^2\left(\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}\right)^2$$

for some $f$ and $T$, the latter of which is the analogue of a string modular parameter. The reason it appears is that a reparametrisation cannot measure the intrinsic length of the worldline, $T := \int_0^1 \mathrm{d}\xi\sqrt{g}$, which is therefore another parameter on which $g$ can depend. The Jacobian for the change of variables from $g$ to $f$ and $T$ is [2]

$$\mathcal{D}g = \frac{\mathrm{d}T}{\sqrt{T}}\,\mathrm{Det}^{1/2}\left(-\frac{1}{T^2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\right)\mathcal{D}f.$$

We can now evaluate the path integral (2.17) by using the reparametrisation invariance to set $g = T^2$, constant, and then integrate over $T$,

$$G_0(x_f; x_i) = \int_0^\infty \frac{\mathrm{d}T}{\sqrt{T}}\mathrm{Det}^{1/2}\left(-\frac{1}{T^2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\right)\int \mathcal{D}x^\mu\; e^{i\int_0^1 \mathrm{d}\xi \dot{x}^2/T + m^2 T}$$

The above determinant is computed with Dirichlet boundary conditions and can be zeta function regulated,

$$\mathrm{Det}^{1/2}\left(-\frac{1}{T^2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\right) = \left(\prod_{n=1}\frac{\pi^2 n^2}{T^2}\right)^{1/2} \rightarrow \mathrm{const.}\sqrt{T}.$$

Splitting $x^\mu$ into classical and quantum pieces, the integral over $x^\mu$ is over the quantum piece which has the Fourier expansion

$$\sum_{m=1} x^\mu_m \sin(m\pi\xi)\sqrt{\frac{2}{T}}.$$

Each $x$ integration results in the inverse of the previous determinant. To do the final integration we can rotate the contour $T \rightarrow -iT$ and we obtain the integral of the heat kernel $\mathcal{K}$ of the Laplacian (with normalisation $\mathcal{K}(T=0) = \delta^{D+1}(x_f - x_i)$),

$$\begin{aligned} G_0(x_f; x_i) &= i\int_0^\infty \frac{\mathrm{d}T}{(4\pi T)^{(D+1)/2}}\, e^{-\frac{1}{4T}(x_f - x_i)^2 + m^2 T} \\ &= i\int \frac{\mathrm{d}^{D+1}k}{(2\pi)^{D+1}}\,\frac{e^{ik\cdot(x_f - x_i)}}{k_\mu k^\mu - m^2}. \end{aligned}$$

The derivation of the gluing property begins with the simple observation that paths from $(\mathbf{x}_1, t_1)$ to $(\mathbf{x}_2, t_2)$ must cross the plane at time $t$ at least once if $t_2 > t > t_1$. This implies the sum over paths defining the propagator can factorised so that formally

$$\sum_{\text{paths AB}} e^{-\text{length(AB)}} = \sum_{\text{C}} \Bigg( \sum_{\text{paths AC}} e^{-\text{length(AC)}} \Bigg) \Bigg( \sum_{\text{paths CB}} e^{-\text{length(CB)}} \Bigg) \tag{2.18}$$

where $C$ lies in the plane at time $t$. To make this factorisation explicit, insert into (2.17) a resolution of the identity,

$$G_0 = \int \mathcal{D}x \left[ \int \mathrm{d}\xi' \sqrt{g(\xi')}\, J(x^0) \delta\left(x^0(\xi') - t\right) \right] e^{iS[x]}. \tag{2.19}$$

For $t_2 > t > t_1$ the delta function always has support on the worldline. The Jacobian $J$ which makes the insertion unity is easily found to be

$$J = \left. \frac{\dot{x}^0(\xi')}{\sqrt{g(\xi')}} \right|_{x^0(\xi')=t},$$

which is reparametrisation invariant. Taking the integral over $\xi'$ outside and distinguishing between worldline times earlier and later than $\xi'$, the path integral is

$$\int \mathrm{d}\xi' \sqrt{g(\mathbf{x}')} \int \left[ \prod_{\xi<\xi'} \mathrm{d}x^\mu(\xi) \right] \mathrm{d}^D\mathbf{x}(\xi') \frac{\dot{x}^0(\xi')}{\sqrt{g(\xi')}} \left[ \prod_{\xi>\xi'} \mathrm{d}x^\mu(\xi) \right]$$
$$\exp\left( i \sum_{\xi<\xi'} S[x(\xi)] + \sum_{\xi>\xi'} S[x(\xi)] \right).$$

We can write $\dot{x}^0(\xi')$ as a two sided derivative

$$\dot{x}(\xi')^0 = \lim_{h\to 0} \frac{x^0(\xi'+h) - x^0(\xi'-h)}{2h}$$

which splits the path integration into a pair of terms each with an insertion. The integrals are invariant under reparametrisations of the worldline and so have no explicit $\xi'$ dependence, the integral over which gives a finite volume, leaving

$$\begin{aligned} G_0 = \frac{1}{2} \int \mathrm{d}^D\mathbf{y} \int \mathcal{D}x^\mu \frac{\dot{x}^0(\xi_{\text{final}})}{\sqrt{g(\xi_{\text{final}})}} e^{iS[x]} \int \mathcal{D}x^\mu\, e^{iS[x]} \\ + \frac{1}{2} \int \mathrm{d}^D\mathbf{y} \int \mathcal{D}x^\mu\, e^{iS[x]} \int \mathcal{D}x^\mu \frac{\dot{x}^0(\xi_{\text{initial}})}{\sqrt{g(\xi_{\text{initial}})}} e^{iS(x)}, \end{aligned} \tag{2.20}$$

where the integral over $\mathbf{y}$ is over the boundary spatial co-ordinate data. The path integrals can be done in the Polyakov approach, and from an integration by parts in the action of (2.17) the insertion of the Jacobian $J$ can be taken outside the integral as a derivative with respect to boundary data,

$$\frac{\partial}{\partial t_{\text{final}}} = \frac{i}{2} \frac{\dot{x}^0(\xi_{\text{final}})}{\sqrt{g(\xi_{\text{final}})}}$$

or with a minus sign for the initial time, giving us the factorisation of the propagator

$$G_0(\mathbf{x}_f, t_f; \mathbf{x}_i, t_i) = -i \int \mathrm{d}^D\mathbf{y}\, G_0(\mathbf{x}_f, t_f, \mathbf{y}, t) \frac{\overleftrightarrow{\partial}}{\partial t} G_0(\mathbf{y}, t, \mathbf{x}_i, t_i), \qquad t_2 > t > t_1. \tag{2.21}$$

The double headed arrow is shorthand for $\rightarrow - \leftarrow$. The explicit calculation is easiest using the Fourier representation. Consider

$$\int \mathrm{d}^D\mathbf{y}\ G_0(\mathbf{x}_2, t_2; \mathbf{y}, t) \frac{\partial}{\partial t} G_0(\mathbf{y}, t, \mathbf{x}_1, t_1)$$

$$= \int \frac{\mathrm{d}^D(\mathbf{p}\,\mathbf{q}, \mathbf{y})\, \mathrm{d}(p_0, q_0)}{(2\pi)^{2(D+1)}} \left(-iq_0\right) \frac{e^{i[p_0(t_2-t) - \mathbf{p}(\mathbf{x}_2-\mathbf{y}) + q_0(t-t_1) - \mathbf{q}(\mathbf{y}-\mathbf{x}_1)]}}{(p_0^2 - E^2(\mathbf{p}))(q_0^2 - E^2(\mathbf{q}))}$$

$$= \int \frac{\mathrm{d}^D\mathbf{p}\, \mathrm{d}(p_0, q_0)}{(2\pi)^{(D+2)}} \left(-iq_0\right) \frac{e^{i[p_0(t_2-t) + q_0(t-t_1) - \mathbf{p}(\mathbf{x}_2-\mathbf{x}_1)]}}{(p_0^2 - E^2(\mathbf{p}))(q_0^2 - E^2(\mathbf{p}))}$$

which follows from doing the $\mathbf{y}$ integration giving $(2\pi)^D\delta\left(\mathbf{p} - \mathbf{q}\right)$ (the usual $i\varepsilon$ prescription is understood but not written explicitly). The $q_0$ integration depends on the sign of $t - t_1$, $\mathrm{Sg}(t - t_1)$, giving

$$\frac{i}{2}\mathrm{Sg}(t - t_1) \int \frac{\mathrm{d}^D\mathbf{p}\, \mathrm{d}p_0}{(2\pi)^{D+1}} \frac{e^{ip_0(t_2-t) - i\mathbf{p}(\mathbf{x}_2-\mathbf{x}_1) - iE(\mathbf{p})|t-t_1|}}{p_0^2 - E^2(\mathbf{p})}.$$

The $p_0$ integration gives

$$\frac{i}{2}\mathrm{Sg}(t - t_1) \int \frac{\mathrm{d}^D\mathbf{p}}{(2\pi)^D} \frac{e^{-i\mathbf{p}(\mathbf{x}_2-\mathbf{x}_1) - iE(\mathbf{p})(|t_2-t| + |t-t_1|)}}{2E(\mathbf{p})}.$$

The leading factors are taken care of if we include the second term in (2.21), the result of which follows. We can write this with the insertion acting on a single propagator by replacing

$$\frac{\overleftrightarrow{\partial}}{\partial t} \longrightarrow -2\frac{\partial}{\partial t}$$

and the following additional cases are immediate,

$$\int \mathrm{d}^D\mathbf{y}\; G_0(\mathbf{x}_2,t_2;\mathbf{y},t)\left(-2\frac{\partial}{\partial t}\right)G_0(\mathbf{y},t;\mathbf{x}_1,t_1) = \begin{cases} iG_0(\mathbf{x}_2,t_2;\mathbf{x}_1,t_1) & t_2>t>t_1 \\ -iG_0(\mathbf{x}_2,t_2;\mathbf{x}_1,t_1) & t_1>t>t_2 \\ iG_I(\mathbf{x}_2,t_2;\mathbf{x}_1,t_1) & t>t_1,t_2 \\ -iG_I(\mathbf{x}_2,t_2;\mathbf{x}_1,t_1) & t<t_1,t_2 \end{cases} \tag{2.22}$$

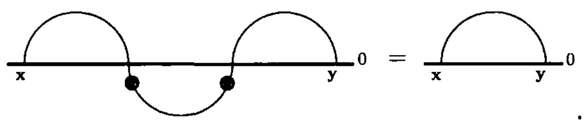
where $G_I$ is an "image propagator" equal to the free space propagator for the points $(\mathbf{x}_2,t_2)$ and the reflection of $(\mathbf{x}_1,t_1)$ in the plane at time $t$. In short, if the two points are on opposite sides of the plane at time $t$, the two propagators are glued to form the usual propagator, if they are on the same side gluing produces the image propagator.

This result should not be confused with the self-reproducing property of heat-kernels (which will be apparent when we discuss strings and which fails when modular parameters are present) but plays a nonetheless fundamental role in field theory. For example, applying it twice gives

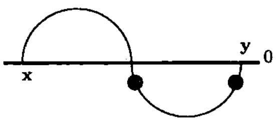
$$\int \mathrm{d}^D\mathbf{x}_3\,\mathrm{d}^D\mathbf{x}_2\; G_0(\mathbf{x}_4,t_4,\mathbf{x}_3,t_3)\left(4\frac{\partial^2}{\partial t_3\,\partial t_2}G_0(\mathbf{x}_3,t_3,\mathbf{x}_2,t_2)\right)G_0(\mathbf{x}_2,t_2,\mathbf{x}_1,t_1)$$

$$= G_0(\mathbf{x}_4,t_4,\mathbf{x}_1,t_1) \quad \text{for} \quad t_4>t_3>t_2>t_1. \tag{2.23}$$

Taking all the $t_i$ to zero gives a useful relation which may be expressed as

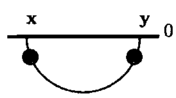



Thus




$= \delta^D(\mathbf{x}-\mathbf{y})$ . (2.24)

From this we deduce that the inverse of the free space propagator at equal time is




. (2.25)

Recall that this expression appeared in the free field vacuum wave functional, (2.15). This is not coincidence, since the vacuum functional must generate the propagator at equal time as a vacuum expectation value,

$$G_0(\mathbf{x},0;\mathbf{y},0) = \int \mathcal{D}\varphi\ \varphi(\mathbf{x})\varphi(\mathbf{y})\Psi_0^2[\varphi], \tag{2.26}$$

and this is a Gaussian integral which (2.24) precisely describes. This is essentially how we wish to describe the free field Schrödinger functional, as the unique object which gives the correct time dependence of the two point function. In the following three sections we will describe time dependence in free field theory using the gluing property.

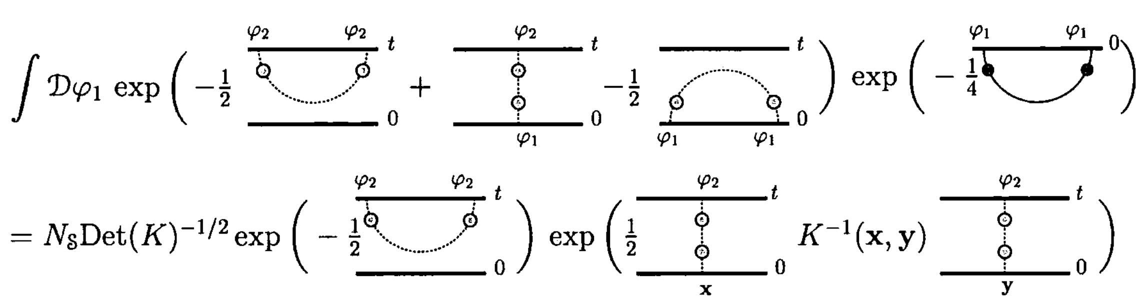
## 2.3 Time evolution of the vacuum

The vacuum wave functional is an eigenstate of the Hamiltonian. Using this and the Schrödinger functional we have two expressions for the vacuum state at time $t$,

$$\Psi_0[\varphi_2,t] = e^{-iE_0t}\Psi_0[\varphi_2,0] = \int \mathcal{D}\varphi_1\, \mathcal{S}[\varphi_2,t;\varphi_1,0]\Psi_0[\varphi_1,0] \tag{2.27}$$

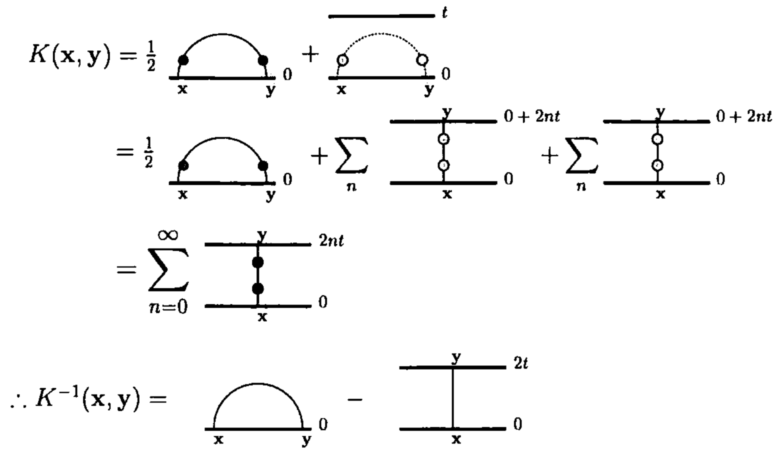
(we will set $\hbar = 1$ for the remainder of this chapter since all our calculations are necessarily of the same order.) The free field integral is Gaussian and can of course be carried out directly, but we will represent it using the diagram expansion. The integral is

$$\int \mathcal{D}\varphi_1 \mathcal{S}[\varphi_2,t;\varphi_1,0]\,\Psi_0[\varphi_1;0] = N_{\mathcal{S}}\times$$



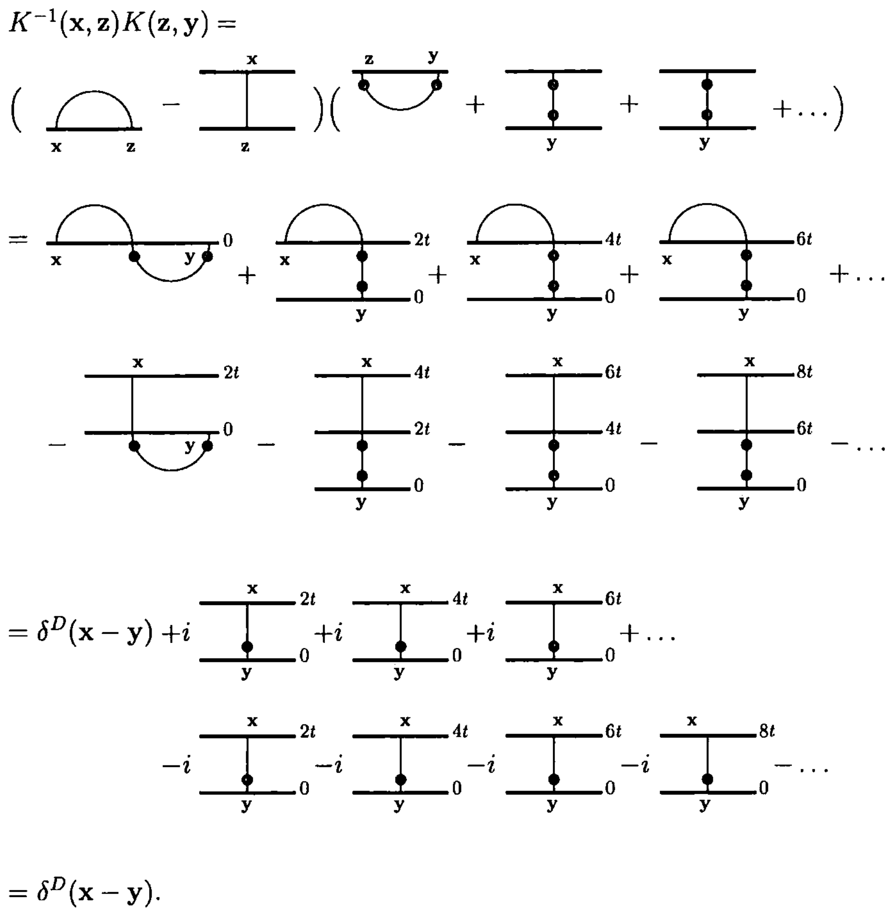

(2.28)

In the second line, the first three terms are the Schrödinger functional, the final term is the vacuum. The Gaussian integral in $\varphi_1$ gives the third line, where the symmetric operator $K$ and its inverse are defined by



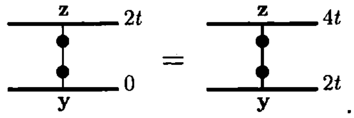

(2.29)

In the definition of $K$, the derivatives on the propagator lead to all the images entering with the same sign (plus). We can check explicitly that the inverse is correct, using the gluing properties (2.22).

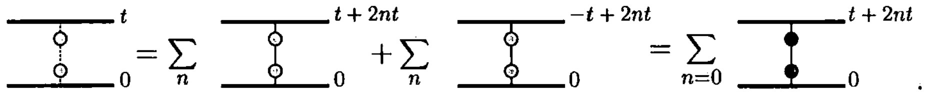



(2.30)

To understand the terms in the third line, either recall that gluing propagators which end on the same side of the boundary produces an image propagator, or, as we have illustrated, use the time dependence of the propagator to translate the diagrams. For example,

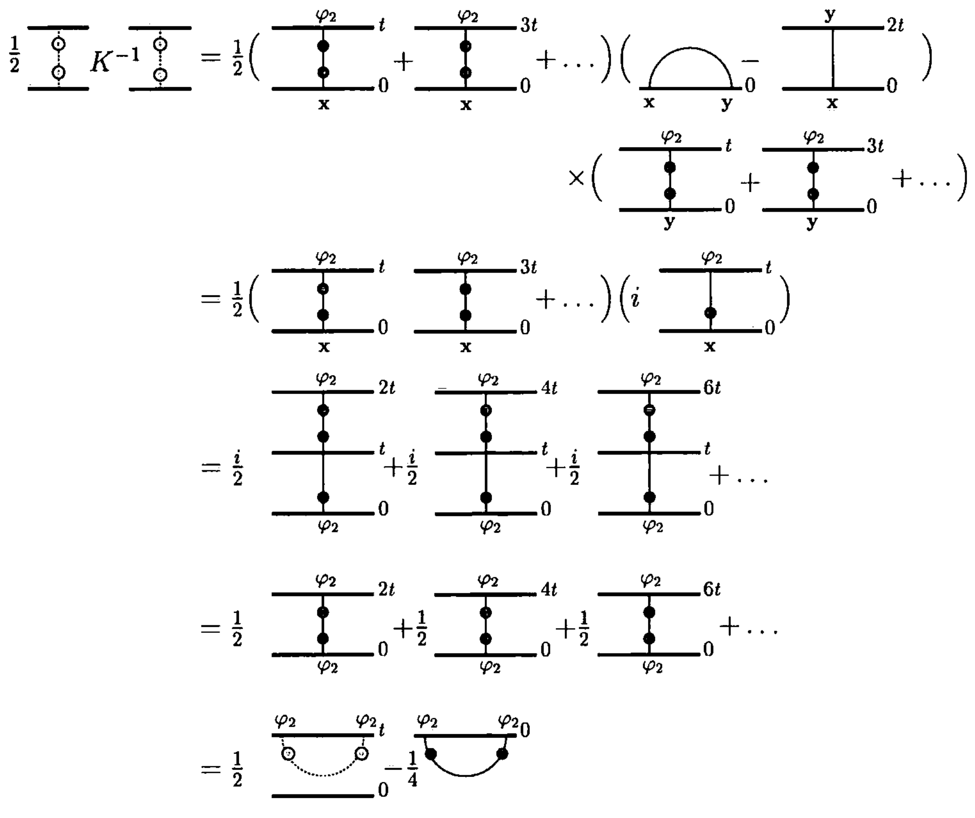


The Schrödinger functional term to be contracted with $K^{-1}$ is



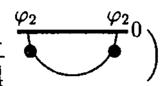
(2.31)

Carrying this out gives the final result of the Gaussian integration,



(2.32)

This removes the remaining $\varphi_2$- dependent term in (2.28) and implies

$$\int \mathcal{D}\varphi_1 \mathcal{S}[\varphi_2, t; \varphi_1, 0]\, \Psi_0[\varphi_1; 0] = N_{\mathcal{S}} \mathrm{Det}(K)^{-1/2} \exp\Big(-\frac{1}{4}$$  $$\Big) \tag{2.33}$$

which is the correct diagram since the vacuum should be time independent, up to a pre factor. From (2.5) we see that the normalisation of the Schrödinger functional is the determinant of the Laplacian with Dirichlet conditions at times 0 and $t$, to the power minus one half. Comparing (2.27) and (2.33) we have an unregulated expression for the free vacuum energy,

$$e^{-iE_0 t} = \mathrm{Det}^{\,-1/2}(\Delta_D)\mathrm{Det}^{\,-1/2}(K). \tag{2.34}$$

The indices in all our calculations can be checked using finite dimensional cases such as

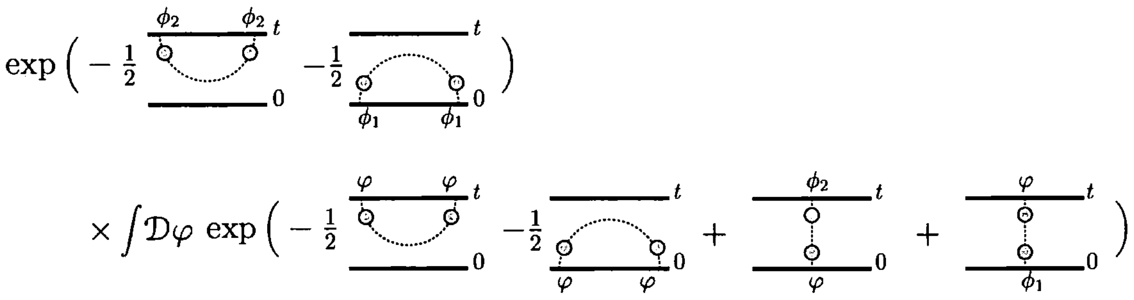
$$\int \prod_k \mathrm{d}u_k \; e^{-\frac{1}{2}u_i A_{ij} u_j + v_j B_{ji} u_i} = \mathrm{Det}\,(A)^{-1/2} e^{\frac{1}{2} v_k B_{ki} A^{-1}_{ij} v_m B_{mj}}. \tag{2.35}$$

## 2.4 Self reproduction of the Schrödinger functional

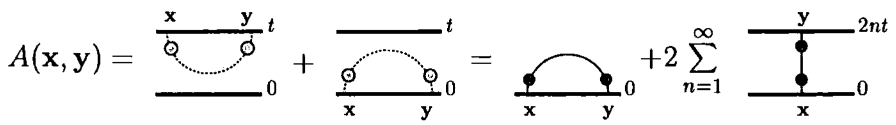
We now recover the self reproducing property of the Schrödinger functional. That is, taking two copies of the functional and integrating over the shared field argument returns the Schrödinger functional for the sum of the times of the initial functionals,

$$\begin{aligned} \int \mathcal{D}\varphi \; \mathbb{S}[\phi_2, t; \varphi, 0]\mathbb{S}[\varphi, t; \phi_1, 0] &= \langle\, \phi_2 \,| e^{-iHt} \int \mathcal{D}\varphi \;|\, \varphi \,\rangle\langle\, \varphi \,| e^{-iHt} |\, \phi_1 \,\rangle \\ &= \mathbb{S}[\phi_2, 2t; \phi_1, 0]. \end{aligned} \tag{2.36}$$

The integral is

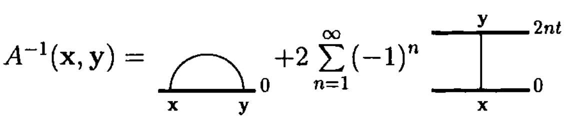


The Gaussian operator, $A$, which we need to invert is now



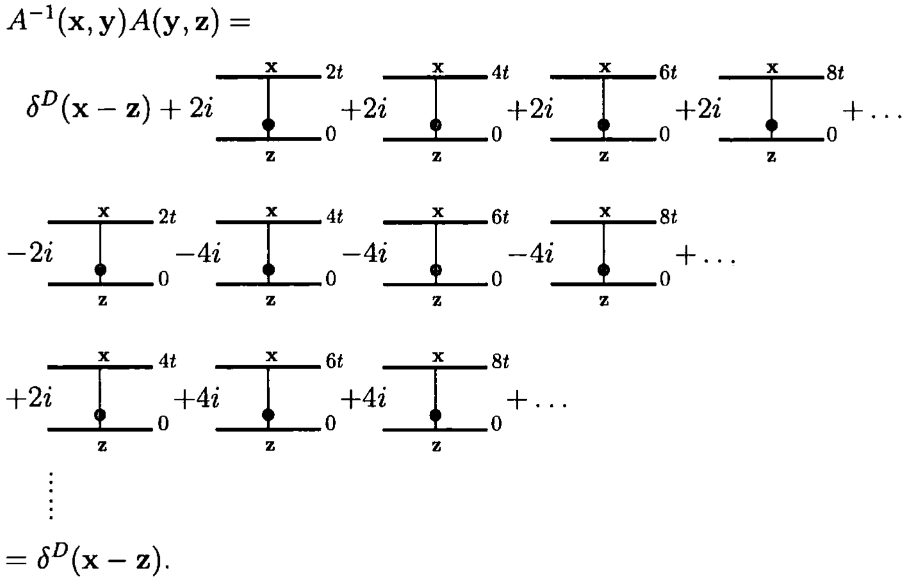
(2.37)

The inverse can be built term by term. Unlike the previous operator $K$, the result is an infinite sum of Feynman diagrams,



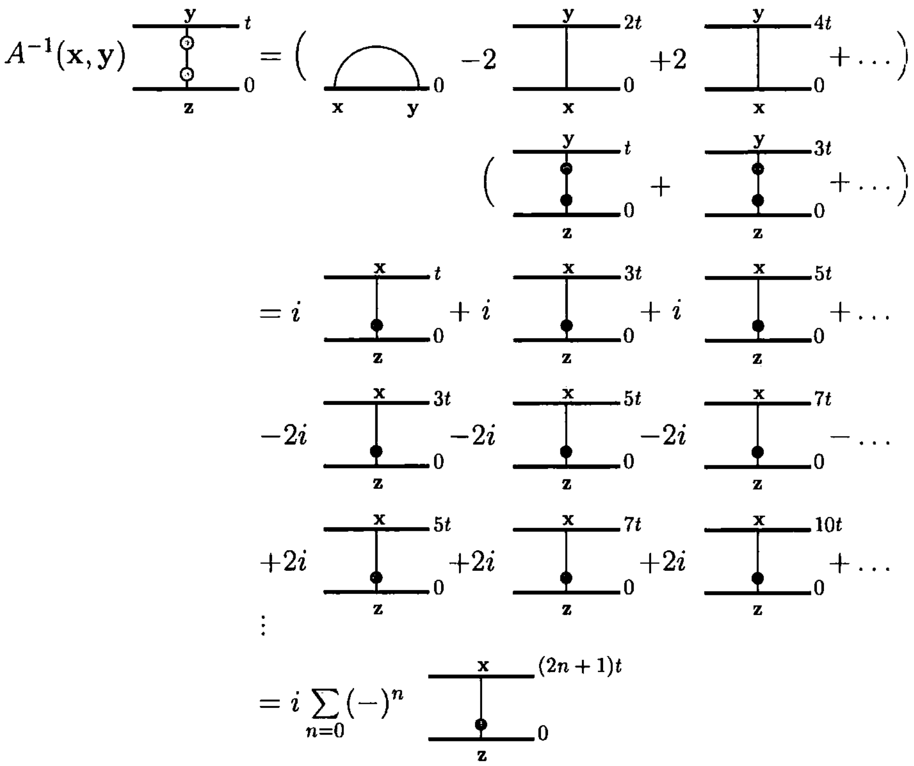
(2.38)

Explicitly,



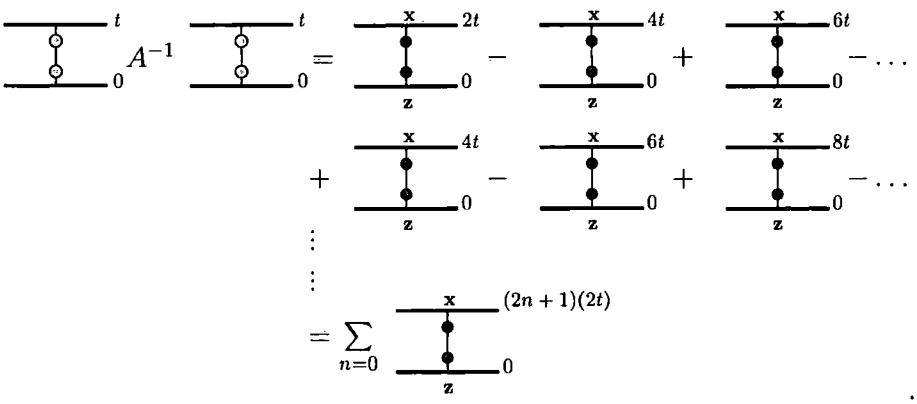
(2.39)

The result of the Gaussian integration is calculated using



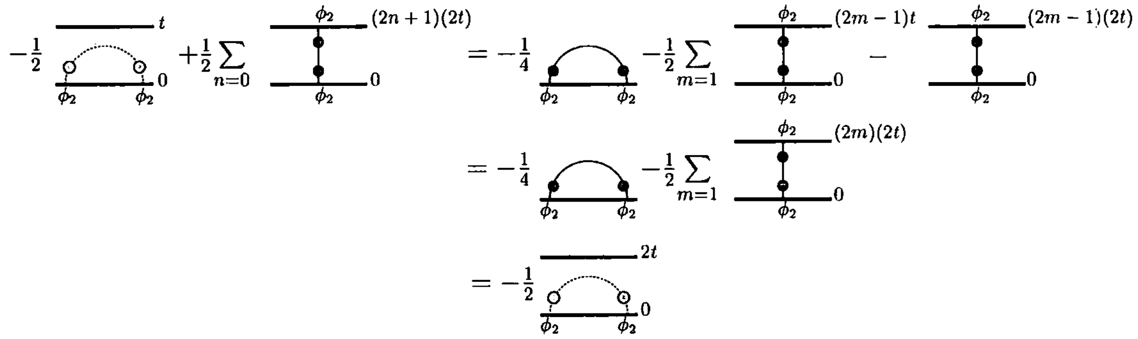
(2.40)

which implies




(2.41)

We now combine this with the remainder of the original Shrödinger functional. We clearly arrive at the correct term connecting $\phi_2$ and $\phi_1$ in $\mathcal{S}[\phi_2, 2t; \phi_1, 0]$. The term quadratic in $\phi_2$ is (similarly for the term quadratic in $\phi_1$)




(2.42)

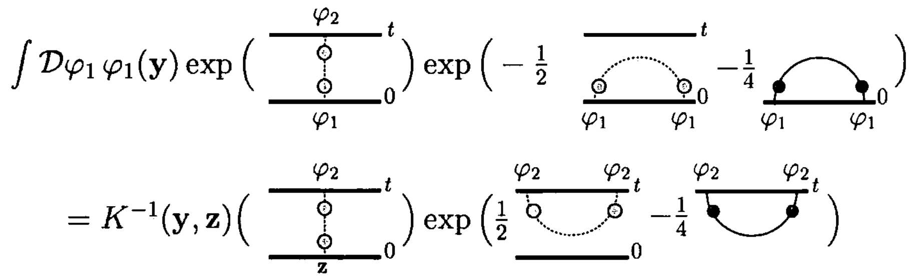
which is the term we expect in $\mathcal{S}[\phi_2, 2t; \phi_1, 0]$.

## 2.5 Time dependence of the two-point function

Finally we apply our methods to an explicitly time dependent object. The two point function can be written in terms of the Schrödinger functional and the vacuum wave functional,

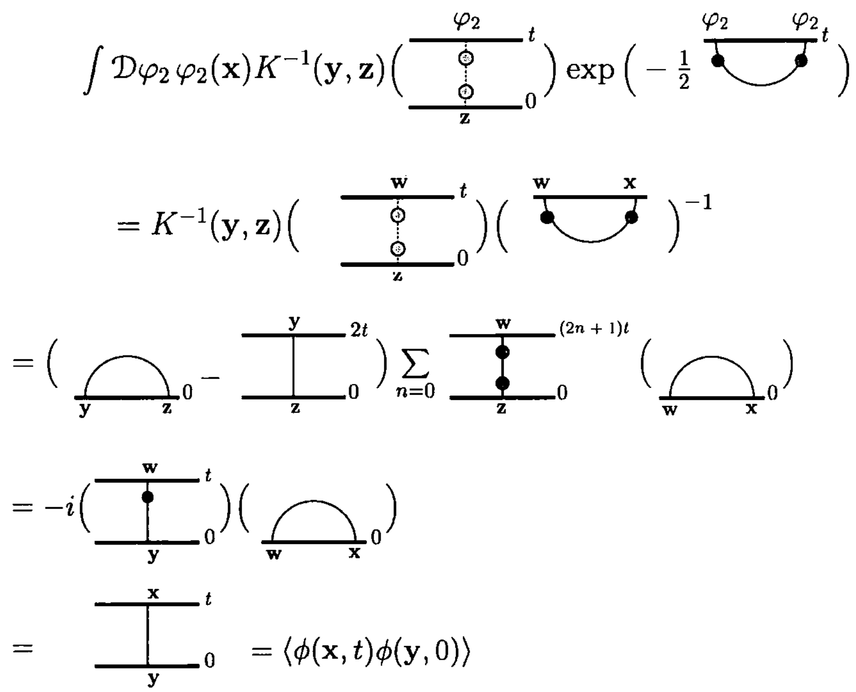
$$\langle \phi(\mathbf{x}, t)\phi(\mathbf{y}, 0)\rangle = \int \mathcal{D}(\varphi_2, \varphi_1)\Psi_0[\varphi_2]\,\varphi_2(\mathbf{x})\ \mathcal{S}[\varphi_2, t; \varphi_1, 0]\,\varphi_1(\mathbf{y})\ \Psi_0[\varphi_1]. \tag{2.43}$$

In the free theory the $\varphi_1$ integral is



(2.44)

The exponential terms are the same as for the evolution of the vacuum, but the insertion of $\varphi_1(\mathbf{y})$ brings down the leading factor. Again the indices can be checked by comparison with the finite dimensional case. The remaining integral over $\varphi_2$ ties together $\mathbf{x}$ and $\mathbf{y}$ giving us



(2.45)

Using the gluing property alone we have shown that equations (2.8) and (2.13) for the Schrödinger functional leads to the correct result for various time dependencies in field theory, in particular for the two point function at unequal times. This argument is invertible; if we know the two point function we can construct the Schrödinger functional and vacuum functional provided the gluing property holds. If we can generalise the gluing property to string theory we can repeat the diagrammatic arguments and construct second quantised string functionals.

We will do just this in the next chapter, and we will work in Euclidean space where the Polyakov functional integrals are better defined. We must verify that our methods work in this representation, and identify any changes from the Lorentzian theory. Extensions of our results to non-flat spacetimes are discussed in the appendices.

## 2.6 Euclidean results

Rotating $x^0 \to -ix^0$ the Euclidean action for a scalar field of mass $m$ is

$$S_E[\phi] = \frac{1}{2}\int \mathrm{d}^\mu x\ \dot{\phi}^2 + \nabla\phi.\nabla\phi + m^2\phi^2. \tag{2.46}$$

The Euclidean free field propagator is

$$\begin{aligned} G_0(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1) &= \int_0^\infty \mathrm{d}T\ \frac{1}{(4\pi T)^{(D+1)/2}} e^{-(x_f - x_i)^2/4T - m^2 T} \\ &= \int \frac{\mathrm{d}^{D+1}k}{(2\pi)^{D+1}} \frac{e^{-ik\cdot(x_f - x_i)}}{k_i k_i + m^2} = \int \frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D} \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}_f - \mathbf{x}_i)}}{2E(\mathbf{k})} e^{-E(\mathbf{k})|t_f - t_i|}. \end{aligned} \tag{2.47}$$

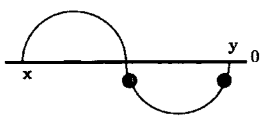
In Euclidean space the propagator factorises as

$$G_0(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1) = \int \mathrm{d}^D\mathbf{x}\, G_0(\mathbf{x}_2, t_2; \mathbf{x}, t) \frac{\overleftrightarrow{\partial}}{\partial t} G_0(\mathbf{x}, t; \mathbf{x}_1, t_1) \qquad t_2 > t > t_1, \tag{2.48}$$

from which follow the gluing properties

$$\int \mathrm{d}^D\mathbf{y}\ G_0(\mathbf{x}_2, t_2; \mathbf{y}, t) \left(-2\frac{\partial}{\partial t}\right) G_0(\mathbf{y}, t; \mathbf{x}_1, t_1) = \begin{cases} G_0(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1) & t_2 > t > t_1 \\ -G_0(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1) & t_1 > t > t_2 \\ G_I(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1) & t > t_1, t_2 \\ -G_I(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1) & t < t_1, t_2. \end{cases} \tag{2.49}$$

The only differences between these rules and those in Minkowski space is a now missing factor of $i$. It follows that the inverse of the propagator at equal time picks up a minus sign,




$$= -\delta^D(\mathbf{x} - \mathbf{y}) \tag{2.50}$$

In Euclidean space the vacuum is given by applying the imaginary time evolution operator $\exp(-\hat{H}t)$ to any state $|\,v\,\rangle$, not orthogonal to the vacuum. For large times

$$\exp(-\hat{H}t)|\,v\,\rangle \sim |\,0\,\rangle e^{-E_0 t}\langle\,0\,|\,v\,\rangle \quad (t \to \infty)$$

where $E_0$ is the energy of the vacuum, and the larger energy eigenvalues are exponentially damped. The path integral representation now follows as before,

$$\Psi_0[\phi] = \int \mathcal{D}\varphi \, \exp\left( - S_E[\varphi] - \int \mathrm{d}^D\mathbf{x}\, \dot{\varphi}(\mathbf{x},0)\phi(\mathbf{x})\right)\Bigg|_{\varphi(\mathbf{x},0)=0}$$

with the action evaluated on the half space $t < 0$. The Schrödinger functional has a similar expression,

$$\mathcal{S}[\phi_2, t_2; \phi_1, t_1] = \int \mathcal{D}\varphi \, \exp\left( -S_E[\varphi] - \int \mathrm{d}^D\mathbf{x}\, \dot{\varphi}(\mathbf{x},t_2)\phi_2 + \int \mathrm{d}^D\mathbf{x}\, \dot{\varphi}(\mathbf{x},t_1)\phi_1\right)\Bigg|_{\varphi(t_1)=0}^{\varphi(t_2)=0}.$$

All of the calculations we have presented can be repeated in Euclidean space, but the only differences to keep track of are minus signs. This is reassuring, as it is the Euclidean results which we will now generalise to string field theory.

# Chapter 3

# Free string field theory

In this chapter we generalise the gluing property of quantum field theory to string field theory using the Polyakov integral description of the string field propagator. This involves using a new description of the ghost sector to represent the infinite number of time derivatives necessary to sew the off–shell worldsheets together. We will begin with a summary of the properties of the string field propagator, then present the new ghost variables and prove the gluing properties. We end with some explicit examples of gluing calculations and the BRST quantisation which defines our ghost structure.

## 3.1 The string field propagator

Off-shell amplitudes are associated with the Polyakov integral on worldsheets with boundaries. The open (closed) string field propagator is the transition amplitude $G(X_f;X_i)$ between arbitrary spacetime curves $X_i^\mu(\sigma)$ and $X_f^\mu(\sigma)$ which bound a finite strip (cylinder),

$$G(X_f;X_i) = \int \mathcal{D}(X,g)\ e^{-\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\,g^{ab}\partial_a X^\mu \partial_b X_\mu}\Bigg|_{X=X_i(\sigma).}^{X=X_f(\sigma)} \tag{3.1}$$

An arbitrary metric on the strip (cylinder) can be written as a diff$\times$Weyl transformation (orthogonal to the conformal killing vector $V=\partial/\partial\sigma$ for the closed string) of a reference metric $\hat{g}_{ab}(T)$ for some value of the modular parameter $T$. A hat on an object indicates it is computed with this metric. The propagator is [40] [41] [42] [45]

$$G_{t_f-t_i}(\mathbf{X}_f;\mathbf{X}_i)=\int_0^\infty \mathrm{d}T\,\mathrm{Jac}(T)(\mathrm{Det}'\hat{P}^\dagger\hat{P})^{\frac{1}{2}}(\mathrm{Det}\,\hat{\Delta})^{-13}\int \mathcal{D}\xi\, e^{-S_{\mathrm{cl}}[X_{\mathrm{cl}},\hat{g}(T)]}. \tag{3.2}$$

The remaining $\xi$ integral is over reparametrisations of the boundary data. The measure on moduli space is $\mathrm{Jac}(T)$ given by

$$\mathrm{Jac}(T)_{\mathrm{open}}=\frac{(\hat{h}_{ab}|\hat{\chi}_{ab})}{(\hat{h}_{ab}|\hat{h}_{ab})^{1/2}},\quad \mathrm{Jac}(T)_{\mathrm{closed}}=\frac{(\hat{h}_{ab}|\hat{\chi}_{ab})}{(\hat{V}|\hat{V})^{1/2}(\hat{h}_{ab}|\hat{h}_{ab})^{1/2}}$$

where $\hat{h}_{ab}$ is the zero mode of $\hat{P}^\dagger$, $\hat{\chi}_{ab}$ is the symmetric traceless part of $\hat{g}_{ab,T}$ and $V=\partial/\partial\sigma$. $X_{\mathrm{cl}}$ satisfies the wave equation in metric $\hat{g}$ with boundary conditions $X^\xi_{\mathrm{cl}}|_{\tau=0}=X_i$, $X^\xi_{\mathrm{cl}}|_{\tau=1}=X_f$ in some parameterisation given by $\xi$. The open string boundary conditions are $X'=0$ at $\sigma=0,\pi$.

If we attach reparametrisation invariant functionals $\Pi_i[\mathbf{X}_i]$, $\Pi_f[\mathbf{X}_f]$ to the boundaries of the worldsheet then this integral can be done trivially to give an (infinite) constant factor, for then

$$\begin{aligned}&\int \mathcal{D}(X_f,X_i)\int \mathcal{D}\xi\, e^{-S_{\mathrm{cl}}[X_{\mathrm{cl}},\hat{g}]}\,\Pi_i[X_f]\Pi_f[X_i]\\ &\qquad=\int \mathcal{D}\big(X_{\mathrm{cl}}|_{\tau=1},X_{\mathrm{cl}}|_{\tau=0}\big)\; e^{-S_{\mathrm{cl}}[X_{\mathrm{cl}},\hat{g}]}\,\Pi_f[X_{\mathrm{cl}}|_{\tau=1}]\Pi_i[X_{\mathrm{cl}}|_{\tau=0}]\int \mathcal{D}\xi.\end{aligned} \tag{3.3}$$

The same applies when we sew two worldsheets together, since $G$ itself is reparametrisation invariant. Rather than work with reparametrisation invariant functionals it is customary to represent the metric degrees of freedom by ghosts and then reparametrisation invariance is replaced by BRST invariance. The ghost sector of the functional integral is

$$(\mathrm{Det}\,P^\dagger P)^{\frac{1}{2}}=\int \mathcal{D}(c,b)\,\exp\left(\int \mathrm{d}^2\sigma\sqrt{g}\,b_{ab}(Pc)^{ab}+\int \mathrm{d}^2\sigma\sqrt{g}\,b_{ab}\chi^{ab}\right). \tag{3.4}$$

There is an additional term to the usual ghost action which must be included to make the functional integral non zero and is due to the presence of a zero mode of $P^\dagger$ (A full Faddeev Popov treatment gauge fixing the Weyl invariant quantity $\sqrt{g}g^{ab}$ includes this term automatically, see [11]). Expanding the antighost

$$b_{ab}=b_0\frac{h_{ab}}{(h_{ab}|h_{ab})^{1/2}}+b'_{ab}$$

where $b_0$ is Grassmann odd and $b'$ is constructed from the non-zero eigenvectors of $P^\dagger$ the integral over $b_0$ becomes saturated and reproduces the modular Jacobian,

$$\int \mathcal{D}(c,b)\,\exp\big(-S_{gh}[b,c]\big) = \frac{(h_{ab}|\chi_{ab,T})}{(h_{ab}|h_{ab})^{1/2}} \int \mathcal{D}(c,b')\,e^{-S_{gh}[b',c]}.$$

The ghosts inherit the Alvarez boundary conditions [8], [43]

$$n^a c_a = 0 \quad \text{on } \partial\mathcal{M}, \qquad n^a t^b (Pc)_{ab} = 0 \quad \text{on } \partial\mathcal{M}, \tag{3.5}$$

where $n^a$ and $t^a$ are, respectively, the normal and tangent vectors on the boundary. In the conformal gauge $\hat{g}_{ab}(T) = \mathrm{diag}(1, T^2)$ and the contributions to the propagator integrand are

$$S_{\mathrm{cl}}[X_{cl},\hat{g}] = \frac{1}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}s\, X_{\mathrm{cl}} n^a \partial_a X_{\mathrm{cl}}, \tag{3.6}$$

$$(\mathrm{Det}\,\hat{\Delta})^{-1/2} = \begin{cases} (4\pi T)^{-1/2}\eta(T)^{-1/2} & \text{open} \\ T^{-1/2}\eta(T)-1 & \text{closed,} \end{cases} \tag{3.7}$$

$$(\mathrm{Det}'\hat{P}^\dagger\hat{P})^{1/2} = \begin{cases} (4\pi T)^{1/2}\eta(T) & \text{open} \\ (4\pi T)\,\eta^2(T) & \text{closed,} \end{cases} \tag{3.8}$$

$$\mathrm{Jac}(T) = \begin{cases} (4\pi T)^{-1/2} & \text{open} \\ (4\pi T)^{-1} & \text{closed,} \end{cases} \tag{3.9}$$

where the Dedekind eta function is

$$\eta(T) := e^{-T/12}\prod_{m=1}^{\infty}(1 - e^{-2mT}). \tag{3.10}$$

The determinants have been zeta-function regulated and the normalisation chosen in line with the particle case.

The propagator in the extended state space of the co-ordinates and ghosts should interpolate between boundary configurations of the ghosts also. To fix the values of $c$ and $b$ on the Dirichlet ($\tau = 0, 1$) boundaries delta-functionals can be inserted into the integration (3.4) giving

$$(\mathrm{Det}'\hat{P}^\dagger\hat{P})^{1/2}\prod_{m=1}\exp\left(\frac{\cosh mT}{\sinh mT}(c^i_m b^i_m + c^f_m b^f_m) - \frac{1}{\sinh mT}(c^i_m b^f_m + c^f_m b^i_m)\right) \tag{3.11}$$

where $b^j_m$, $c^j_m$ are the Fourier modes of the initial, final ghost configurations [41].

## 3.2 Corners and sewing for the open string

There is a caveat to these calculations in the case of the open string. We know that the integrand in (3.1) is independent of the Liouville mode when $D+1=26$ [44] at least for worldsheets without boundaries, giving a Weyl invariant string theory. Although our worldsheet is topologically a disk, we have boundaries with marked points where boundary conditions change – the corners. This generates extra contributions to the Weyl dependence of the determinants in (3.2) as we now describe.

As an example consider the determinant of the Laplacian. Under an infinitesimal Weyl scaling $\rho \to \rho + \delta\rho$ the variation of the determinant is, using the usual heat kernel regularisation [9],

$$\delta \log \mathrm{Det}\, \Delta = -\int \mathrm{d}^2\sigma \sqrt{g}\ \mathcal{K}(\sigma, \sigma, \tau, \tau; \epsilon)\delta\rho(\sigma). \tag{3.12}$$

We know that the variation must have an expansion in powers of the cutoff of form

$$\begin{aligned}\delta \log \mathrm{Det}\, \Delta =& -\frac{1}{24\pi}\int \mathrm{d}^2\sigma\sqrt{g}\ R\,\delta\rho - \frac{1}{4\pi\epsilon}\int \mathrm{d}^2\sigma\sqrt{g}\,\delta\rho \\ &+ \sum_i A_i \frac{1}{\sqrt{\epsilon}}\int \mathrm{d}s\,\delta\rho + B_i\int \mathrm{d}s\, n^a\partial_a\delta\rho + C_i\int \mathrm{d}s\ K_g\delta\rho \\ &+ E\sum_j \delta\rho(\sigma_j) + \mathcal{O}(\sqrt{\epsilon})\end{aligned} \tag{3.13}$$

where $i$ runs over the boundaries and $j$ over the distinguished points. The divergent volume and surface terms can be removed by local counter-terms in the string action. What remains is removed by the metric integral when $D+1=26$ excluding the contribution from the corners [45], [46]. So even in the critical dimension, the off-shell propagator is not truly Weyl invariant.

We will need to compute these corner anomalies for various determinants. Each corner contributes equally and independently (for the same change in boundary conditions). Since the anomaly is a local effect it is insensitive to the global topology of the worldsheet, so we can work on the simpler geometry of the upper right quadrant with appropriate conditions on the axes. Finally, since the anomaly only depends on a single value of the Liouville field we can compute it using a constant $\delta\rho$.

Returning to the Laplacian the quadrant has Dirichlet conditions on the $x$-axis and Neumann conditions on the $y$-axis. The heat kernel for this geometry is given by the method of images,

$$\begin{aligned}\mathcal{K}(x,y,x',y';\epsilon) &= \mathcal{K}_0(x,y,x',y';\epsilon) - \mathcal{K}_0(x,y,x',-y';\epsilon)\\ &\quad + \mathcal{K}_0(x,y,-x',y';\epsilon) - \mathcal{K}_0(x,y,-x',-y';\epsilon),\end{aligned} \tag{3.14}$$

where the free space heat kernel for the Laplacian is

$$\mathcal{K}_0(x,y,x',y';\epsilon) = \frac{1}{4\pi\epsilon}\exp\left( - \frac{(x-x')^2+(y-y')^2}{4\epsilon}\right). \tag{3.15}$$

The variation in the determinant with a constant Weyl scaling is, writing down only the cutoff independent, non-zero corner anomaly in the final line,

$$\begin{aligned}\delta \log \mathrm{Det}\,\Delta &= -\delta\rho \int_0^\infty \mathrm{d}x\,\mathrm{d}y\ \mathcal{K}(x,y,x,y;\epsilon)\\ &= -\delta\rho \int_0^\infty \mathrm{d}x\,\mathrm{d}y\ \frac{1}{4\pi\epsilon} - \frac{1}{4\pi\epsilon}e^{-y^2/\epsilon} + \frac{1}{4\pi\epsilon}e^{-x^2/\epsilon} - \frac{1}{4\pi\epsilon}e^{-(x^2+y^2)/\epsilon}\\ &= \frac{1}{16}\delta\rho + \dots\end{aligned} \tag{3.16}$$

The corner anomaly is not a sickness of the off-shell theory. It is there to cancel anomalies generated by the process of sewing worldsheets together or sewing string functionals onto Dirichlet sections of worldsheets. When we sew two worldsheets $\mathcal{M}_1$ and $\mathcal{M}_2$ together by integrating over all possible boundary values of $X$ (the ghosts behave similarly, and we will neglect modular parameters for the moment) the classical actions combine to give the classical action on the sewn worldsheet. The Gaussian integration brings down the determinant of $n^a\partial_a$ to the power minus one half, computed with the harmonic extension of the boundary data into the bulk, which sews the determinants together [47];

$$\begin{aligned}(\mathrm{Det}\,\Delta)^{-1/2}_{\mathcal{M}_1}(\mathrm{Det}\,\Delta)^{-1/2}_{\mathcal{M}_2}\int \mathcal{D}X_{\mathrm{bhd}}\, e^{-\int_{\mathcal{M}_1} \mathrm{d}s X_{\mathrm{bhd}} n^a\partial_a X_{cl}}\, e^{-\int_{\mathcal{M}_2} \mathrm{d}s X_{\mathrm{bhd}} n^a\partial_a X_{cl}}\\ = (\mathrm{Det}\,\Delta)^{-1/2}_{\mathcal{M}_1+\mathcal{M}_2} e^{-\int_{\mathcal{M}_1+\mathcal{M}_2} \mathrm{d}s X_{\mathrm{bhd}} n^a\partial_a X_{cl}}.\end{aligned} \tag{3.17}$$

The anomalous boundaries become a bulk piece of the sewn worldsheet, which has no privileged structure and so cannot carry the anomaly– how does this come about?

The answer is the determinant of $n^a\partial_a$ has its own corner anomaly. The calculation is given in [46] so we will just state

$$\delta \log \mathrm{Det}\, n^a\partial_a = \mp\frac{1}{8}\delta\rho - \frac{1}{2\pi\epsilon}\int \mathrm{d}s\, \delta\rho \tag{3.18}$$

where the minus (plus) sign is for Neumann (Dirichlet) boundary conditions at $\sigma = 0, \pi$. The open string co-ordinates have Neumann conditions so the anomaly is minus twice that coming from a single Laplacian, (3.16), ensuring that two worldsheets are sewn without anomaly in the bulk.

## 3.3 Sewing worldsheets

The appearance of the corner anomaly has led us naturally to sewing. In this section we will prove the extension of the gluing property (2.48) to the string propagator as a method of sewing worldsheets. The details are a little involved, but in the next section we will give explicit examples.

If we represent the propagator as an amplitude

$$G(X_f, b^f, c^f; X_i, b^i, c^i) = \int_0^\infty \mathrm{d}T\ \langle\, X_f, b^f, c^f\, | e^{-\hat{H}T} |\, X_i, b^i, c^i\,\rangle \tag{3.19}$$

then the standard sewing prescription discussed above, integrating over all boundary values of $X^\mu$ and the ghosts shared between the two propagators, returns

$$\int_0^\infty \mathrm{d}U \int_0^\infty \mathrm{d}T\ \langle\, X_f, b^f, c^f\, | e^{-\hat{H}(T+U)} |\, X_i, b^i, c^i\,\rangle, \tag{3.20}$$

which is incorrect, since we have a redundant modular integral which gives an infinite factor. Carlip showed [47] that by inserting the Hamiltonian acting on the boundary of one worldsheet the redundant length parameter is removed and the moduli spaces are correctly sewn. This applies to general worldsheets but is suitably illustrated by sewing two propagators,

$$\begin{aligned}
&\int \mathcal{D}(X,b,c)\ G(X_f,b^f,c^f;X,b,c)\ \hat{H}G(X,b,c;X_i,b^i,c^i)\\
&= -\int \mathcal{D}(X,b,c)\int_0^\infty \mathrm{d}T\ \langle X_f,b^f,c^f\,|e^{-\hat{H}T}|\,X,b,c\rangle \int_0^\infty \mathrm{d}U\ \frac{\partial}{\partial U}\langle X,b,c\,|e^{-\hat{H}U}|\,X_i,b^i,c^i\rangle\\
&= \int \mathcal{D}(X,b,c)\int_0^\infty \mathrm{d}T\ \langle X_f,b^f,c^f\,|e^{-\hat{H}T}|\,X,b,c\rangle\delta\big[X-X_i,b-b^i,c-c^i\big]\\
&= \int_0^\infty \mathrm{d}T\ \langle X_f,b^f,c^f\,|e^{-\hat{H}T}|\,X_i,b^i,c^i\rangle.
\end{aligned} \tag{3.21}$$

The Hamiltonian is the derivative of the second integrand with respect to the modular parameter, which reduces one propagator to a delta functional[1]. Sewing this onto the second propagator trivially produces the desired result.

However, this method is inappropriate for our means. In the Schrödinger representation we are interested in evolving states between successive times so the propagator would be calculated with the particular boundary conditions $X^0$ = constant and we would not integrate over $X^0$ when sewing, just as we did in the previous chapter for quantum field theory. However this would leave behind one Laplacian determinant from each worldsheet, so there is something more going on.

That there should exist a method of sewing worldsheets appropriate to our needs may not be immediately obvious because of the extended nature of the string, but performing a similar trick to (2.19) gives a strong indication; insert a resolution of the identity into the functional integral defining the propagator,

$$G(X_f;X_i) = \int \mathcal{D}(X,g)\left[1 = \int \mathcal{D}C\, J(C,X^0)\delta\left(X^0(\sigma,C(\sigma))-t\right)\right]e^{-S_E(X,g)} \tag{3.22}$$

where the new integral is over curves $C$ on the worldsheet. For $t_f > t > t_i$ the delta functional has support on the worldsheet, and the Jacobian is

$$\prod_\sigma \dot{X}^0(\sigma,C_X(\sigma)) \tag{3.23}$$

[1]Strictly speaking this is only true for a positive definite Hamiltonian, so for the bosonic string there is the tachyon divergence to be regulated.

where $C_X$ is the curve on which $X^0(\sigma, C_X(\sigma)) = t$. We will now describe how this infinite product of time derivatives can be represented and used to generalise (2.48). The key is the metric integral. Taking the Alvarez boundary conditions in the conformal gauge, a basis of reparametrisations $\xi^a = (\xi^\sigma, \xi^\tau)$ on the strip (with metric as given earlier, so $\tau \in [0,1]$ and $\sigma \in (0,\pi)$ is given by

$$\xi^1_{nm} = N_m \begin{pmatrix} \cos(m\pi\tau)\sin(n\sigma) \\ 0 \end{pmatrix}, \quad \xi^2_{nm} = N_m \begin{pmatrix} 0 \\ \sin(n\pi\tau)\cos(m\sigma) \end{pmatrix} \tag{3.24}$$

with normalisation $N_m = 2^{1-\frac{1}{2}\delta_{m=0}}/\sqrt{T\pi}$. The reparametrisations split into orthogonal pieces, with the set $\{\xi^2\}$ obeying

$$n^a\xi^2_a = t^a\xi^2_a = \partial_\sigma\xi^2_\sigma = 0$$

on the Dirichlet boundaries – so only half of all reparametrisations couple to these boundaries. If we were to compute the propagator with conditions on the metric fixing $g_{\sigma\sigma}$, much as we do for the co-ordinates, then to sew two propagators together we would integrate over all values of the boundary metric which would sew together only half of the determinant of $P^\dagger P$. Since $(\mathrm{Det}\, P^\dagger P)^{1/2}$ cancels two copies of $(\mathrm{Det}\,\Delta)^{-1/2}$, this would leave a determinant behind to cancel that from $X^0$.

We now make this rigourous with a proof that the gluing property can be generalised to the string field propagator. The proof in three stages.

1. We will introduce a change of variables in the ghost sector allowing us to realise the discussion following (3.24).

2. We then show that when integrating over the boundary data, the ghosts cancel the unwanted effects of not integrating over $X^0$. This combines the integrands of the propagators into the standard "sewn" form, as in (3.20).

3. Finally, we insert a time derivative between the propagators being sewn and, similarly to Carlip's method, use it to remove the extra modular parameter and contract one propagator to a delta functional, integrating over which returns the sewn propagator.

### 1. Ghosts

The rôle of the reparametrisation ghosts in string theory is to cancel the undesirable effects of including the $X^0$ oscillators. They will do the same thing for us here, although to a different end. Starting with the standard $b-c$ system,

$$(\mathrm{Det}'P^\dagger P)^{1/2} = \int \mathcal{D}(c^a, b'_{ab}) \ \exp\Big( - \int \mathrm{d}^2\sigma\sqrt{g}\ b'_{ab}P(c)^{ab}\Big)$$

we change variables $b' = P\gamma$ and pick up a Jacobian $(\mathrm{Det}\, P^\dagger P)^{-1/2}$ (recall that the Jacobian for a Grassmann change of variables is the inverse of the bosonic Jacobian), which we can represent as a bosonic vector integral. Our new ghost system is therefore

$$\begin{aligned}(\mathrm{Det}'P^\dagger P)^{1/2} = \int \mathcal{D}(c^a, \gamma^a, f^a) \exp\Big(-\frac{1}{2}\int \mathrm{d}^2\sigma\sqrt{g}\ (Pf)_{ab}(Pf)^{ab}\Big)& \\ \times \exp\Big( - \frac{1}{2}\int \mathrm{d}^2\sigma\sqrt{g}\ (P\gamma)_{ab}(Pc)^{ab}\Big),&\end{aligned} \tag{3.25}$$

where $f^a$ is the bosonic worldsheet vector. We have included a factor of one half in front of the new action for convenience later. The operator $P$ is not invertible on the strip or the cylinder, but this should not worry us since we are merely changing the representation of $\mathrm{Det}'P^\dagger P$, and the eigenvectors and eigenvalues of $b'_{ab}$ and $P\gamma_{ab}$ are in one to one correspondence [8]. However, this change of variable will have an interesting effect on the BRST transformation, as is discussed in the final sections of this chapter.

The propagator in the extended state space is the transition amplitude between arbitrary values of $X^\mu, c^\sigma$, $\gamma^\sigma$, $f^\sigma$. The ghost integrals can be done by expanding around a classical solution satisfying $P^\dagger P = 0$ (or inserting delta functionals as described earlier). The $\tau$–components of the fields, corresponding to the set of reparametrisations $\{\xi^2\}$ discussed earlier, are integrated out. For the open string, the classical ghost fields obey

$$J^\sigma_{\mathrm{cl}}(\sigma,0) = \sum_{m=1} J^i_m \sin(m\sigma)\sqrt{\frac{2}{\pi}}, \qquad J^\sigma_{\mathrm{cl}}(\sigma,1) = \sum_{m=1} J^f_m \sin(m\sigma)\sqrt{\frac{2}{\pi}}$$

$$J^\tau_{\mathrm{cl}} \equiv 0$$

where the field $J \in \{f, c, \gamma\}$. The quantum fields obey

$$
\begin{aligned}
n^a \delta J_a &= t^a \delta J_a = 0 \quad \text{on Dirichlet boundaries,} \\
n^a \delta J_a &= n^a t^b P(\delta J)_{ab} = 0 \quad \text{on Neumann boundaries.}
\end{aligned}
\tag{3.26}
$$

For the closed string the change of variables is only well defined up to shifts proportional to the CKV, $J^a \to J^a + \lambda V^a$ for $J^a \in \{c^a, \gamma^a, f^a\}$, so we choose $(J|V) = 0$ which removes the centre of mass from the classical pieces.

With these boundary conditions the actions separate into a quantum piece giving the determinants,

$$
\int \mathrm{d}^2\sigma \sqrt{g}\, \delta\gamma_a \,(P^\dagger P \delta c)^a + \int \mathrm{d}^2\sigma \sqrt{g}\, \delta f_a \,(P^\dagger P \delta f)^a, \tag{3.27}
$$

and a classical action

$$
\int \mathrm{d}s\, \gamma^\sigma_{cl} \,(P c_{\mathrm{cl}})_{\sigma\tau} + \int \mathrm{d}s\, f^\sigma_{\mathrm{cl}} \,(P f_{\mathrm{cl}})_{\sigma\tau} \tag{3.28}
$$

evaluated at $\tau = 0, 1$. With the given boundary conditions the classical action reduces to that for a single free boson and a pair of free Grassmann fields,

$$
\int_0^\pi \mathrm{d}s\, \gamma^\sigma_{cl}\, \partial_\tau c_{\sigma \mathrm{cl}} + \int_0^\pi \mathrm{d}s\, f^\sigma_{\mathrm{cl}}\, \partial_\tau f_{\sigma \mathrm{cl}} \tag{3.29}
$$

The exponent in the Grassmann part is easily verified to give (3.11) with the modes $b_m$ replaced by the modes $\gamma_m$ of $\gamma^\sigma$.

### 2. Integrating over boundary data

We will again write the propagator as an amplitude,

$$
G(t_f, \mathbf{B}_f; t_i, \mathbf{B}_i) = \int_0^\infty \mathrm{d}T \; \langle\, t_f \,| e^{-H_0 T} |\, t_i \,\rangle \langle\, \mathbf{B}_f \,| e^{-\mathbf{H}T} |\, \mathbf{B}_i \,\rangle, \tag{3.30}
$$

where the states $\langle\, t \,|$ are ordinary quantum mechanical states for the time variable and $\langle\, \mathbf{B} \,|$ represents boundary data for $\mathbf{X}$, $c^\sigma, \gamma^\sigma, f^\sigma$. The contribution from the $X^0$ oscillators ($\eta(T)^{-1/2}$ for the open string, $\eta(T)^{-1}$ for the closed string) precisely cancel against the ghost $\tau$–component contributions and the modular Jacobians. Write the worldsheet Hamiltonian as

$$
H = H_0 + \mathbf{H} \tag{3.31}
$$

with $H_0 = -\frac{\partial^2}{\partial t^2}$ and $\mathbf{H}$ the Hamiltonian for all remaining degrees of freedom. Explicitly,

$$\begin{aligned}\langle\, t_f\,|e^{-H_0 T}|\, t_i\,\rangle &= \frac{1}{\sqrt{T}} e^{-(t_f-t_i)^2/4\pi\alpha' T},\\ \langle\,\mathbf{B}_f\,|e^{-\mathbf{H}T}|\,\mathbf{B}_i\,\rangle &= \frac{1}{T^{12}} e^{-S_{\rm cl}[\mathbf{B}]}\eta(T)^{-24}\end{aligned} \tag{3.32}$$

where $S_{\rm cl}$ is the classical action for $\mathbf{X}$ and the ghosts. It is a simple matter to check that when we integrate over the remaining boundary data shared by two propagators with modular parameters $T$ and $U$ we get back the amplitude for the remaining boundary data with modular parameter $T+U$– the new ghosts make this integration into a resolution of the identity,

$$\int \mathcal{D}\mathbf{B}\ \langle\,\mathbf{B}_f\,|e^{-HT}|\,\mathbf{B}\,\rangle\langle\,\mathbf{B}\,|e^{-HU}|\,\mathbf{B}_i\,\rangle = \langle\,\mathbf{B}_f\,|e^{-H(T+U)}|\,\mathbf{B}_i\,\rangle. \tag{3.33}$$

### 3. Sewing with the time derivative

Taking two propagators, we insert a time derivative between them and integrate over the boundary data as we have described to obtain

$$\int_0^\infty \mathrm{d}T\mathrm{d}U\ \langle\, t_f\,|e^{-H_0 T}|\,t\,\rangle\frac{\overleftrightarrow{\partial}}{\partial t}\langle\, t\,|e^{-H_0 U}|\,t_i\,\rangle\langle\,\mathbf{B}_f\,|e^{-\mathbf{H}(T+U)}|\,\mathbf{B}_i\,\rangle. \tag{3.34}$$

We can write the time derivative as a double derivative and an integral,

$$-\int_t^\infty \mathrm{d}\tilde{t}\int_0^\infty \mathrm{d}T\int_0^\infty \mathrm{d}U\,\langle\, t_f\,|e^{-H_0 T}|\,\tilde{t}\,\rangle\frac{\overleftrightarrow{\partial^2}}{\partial \tilde{t}^2}\langle\,\tilde{t}\,|e^{-H_0 U}|\,t_i\,\rangle\langle\,\mathbf{B}_f\,|e^{-\mathbf{H}(T+U)}|\,\mathbf{B}_i\,\rangle.$$

Using the definition (3.31) this is

$$\int_t^\infty \mathrm{d}\tilde{t}\int_0^\infty \mathrm{d}T\int_0^\infty \mathrm{d}U\left(\frac{\partial}{\partial U}-\frac{\partial}{\partial T}\right)\left\{\langle\, t_f\,|e^{-H_0 T}|\,\tilde{t}\rangle\langle\,\tilde{t}\,|e^{-H_0 U}|\,t_i\,\rangle\langle\,\mathbf{B}_f\,|e^{-\mathbf{H}(T+U)}|\,\mathbf{B}_i\,\rangle\right\}$$

plus two cancelling terms. In the above we see the similarity to Carlip's method – an insertion of the time derivative is related to an insertion of the worldsheet Hamiltonian which gives a derivative with respect to the modular parameter. We can now perform the integral over one modular parameter,

$$\begin{aligned}&\int_t^\infty \mathrm{d}\tilde{t}\int_0^\infty \mathrm{d}U\,\delta\left(t_f-\tilde{t}\right)\langle\,\tilde{t}\,|e^{-H_0 U}|\,t_i\,\rangle\langle\,\mathbf{B}_f\,|e^{-\mathbf{H}U}|\,\mathbf{B}_i\,\rangle\\ &-\int_t^\infty \mathrm{d}\tilde{t}\int_0^\infty \mathrm{d}T\,\delta\left(\tilde{t}-t_i\right)\langle\, t_f\,|e^{-H_0 T}|\,\tilde{t}\rangle\langle\,\mathbf{B}_f\,|e^{-\mathbf{H}T}|\,\mathbf{B}_i\,\rangle,\end{aligned}$$

and for $t_f > t > t_i$ the $\tilde{t}$ integral has support on only one delta function, giving

$$\int_0^\infty \mathrm{d}U \, \langle \, t_f \, | e^{-H_0 U} | \, t_i \, \rangle \langle \, \mathbf{B}_f \, | e^{-\mathbf{H}U} | \, \mathbf{B}_i \, \rangle \equiv G(t_f, \mathbf{B}_f; t_i, \mathbf{B}_i)$$

recovering the propagator. Overall, we have shown that the Euclidean generalisation of (2.21) holds in string theory as

$$\int \mathcal{D}\mathbf{B} \; G(t_2, \mathbf{B}_2; t, \mathbf{B}) \, \frac{\overleftrightarrow{\partial}}{\partial t} \, G(t, \mathbf{B}; t_1, \mathbf{B}_1) = G(t_2, \mathbf{B}_2; t_1, \mathbf{B}_1), \quad t_2 > t > t_1. \qquad (3.35)$$

A proof that the various corollaries to the gluing property given in Section 2.2 also apply to string field theory is given in Appendix B. Uses of Carlip's sewing method extend to quantum field theory (since particle propagators also contain modular integrals), for example it is intimately related to the recurrence relations of MHV rules, see [59].

## 3.4 Discussion

We will now give some explicit examples of sewing worldsheets and check our gluing property using the corner anomaly. A simple and readily computable example is for pointlike initial and final co-ordinate states and vanishing ghost states $c_m = \gamma_m = f_m = 0$. In this case when we sew two closed string propagators the integral over boundary data as we have described returns

$$\int_0^\infty \mathrm{d}(T, U) \frac{1}{\sqrt{T}} e^{-(t_f - t)^2/4T} \left( \frac{\overleftrightarrow{\partial}}{\partial t} \right) \frac{1}{\sqrt{U}} e^{-(t - t_i)^2/4U}$$
$$\times \frac{1}{(T+U)^{25/2}} e^{-(\mathbf{x}_f - \mathbf{x}_i)^2/4(T+U)} e^{T+U} \prod_{m=1} (1 - e^{-2m(T+U)})^{-24}.$$

Expanding the final product gives

$$\sum_m \eta_m \int_0^\infty \mathrm{d}(T, U) \frac{1}{\sqrt{T}} e^{-(t_f - t)^2/4T} \left( \frac{\overleftrightarrow{\partial}}{\partial t} \right) \frac{1}{\sqrt{U}} e^{-(t - t_i)^2/4U}$$
$$\times \frac{1}{(T+U)^{25/2}} e^{-(\mathbf{x}_f - \mathbf{x}_i)^2/4(T+U)} e^{-(2m-1)(T+U)}$$

where the $\eta_m$ are constants. Comparing with (2.47) this is a sum over masses of the particle gluing result, so we have

$$\sum_m \eta_m \int_0^\infty dT \, \frac{1}{T^{13}} e^{-(t_f-t_i)^2/4T-(\mathbf{x}_f-\mathbf{x}_i)^2/4T} e^{-(2m-1)T}$$
$$= \int_0^\infty dT \, \frac{1}{T^{13}} e^{-(t_f-t_i)^2/4T-(\mathbf{x}_f-\mathbf{x}_i)^2/4T} \prod_{m=1} (1-e^{-2mT})^{-24}$$

which is the closed string propagator connecting pointlike initial and final states.

The sewing description we have given may seem surprising given the extended nature of the string. Fortunately, the open string affords us a check. We can verify that the corner anomalies correctly cancel between the determinants of $(\mathrm{Det}' P^\dagger P)^{1/4}$ and $(\mathrm{Det}\,\Delta)^{-1/2}$ when we sew. Under an infinitesimal Weyl transformation the variation of $(\mathrm{Det}' P^\dagger P)$ is [8]

$$\begin{aligned}
\delta \log \mathrm{Det}\, P^\dagger P &= -\int_\epsilon^\infty \frac{ds}{s} \mathrm{Tr}\big[\, \delta\, e^{-sP^\dagger P} \,\big] \\
&= \int_\epsilon^\infty ds \, \mathrm{Tr}\big[\, (-2P^\dagger \delta\rho P + P^\dagger P \delta\rho) e^{-sP^\dagger P} \,\big] \\
&= \int_\epsilon^\infty ds \, \mathrm{Tr}\big[\, (-2\delta\rho P P^\dagger e^{-sPP^\dagger} + \delta\rho P^\dagger P) e^{-sP^\dagger P} \,\big] \\
&= -2\mathrm{Tr}\big[\, \delta\rho e^{-\epsilon P P^\dagger} \,\big] + \mathrm{Tr}\big[\, \delta\rho e^{-\epsilon P^\dagger P} \,\big].
\end{aligned}$$

We have used the cyclicity of the trace in going from the second to the third line. However we know that we can calculate the corner anomaly with a constant $\delta\rho$ in which case the above becomes

$$\delta \log \mathrm{Det}\, P^\dagger P = -\delta\rho \, \mathrm{Tr}\big[\, e^{-\epsilon P^\dagger P} \,\big]. \tag{3.36}$$

For the ghost $\tau$–components the non-zero contribution to the heat kernel for $P^\dagger P$ on the upper right quadrant is, by the method of images,

$$\mathcal{K}^{\tau\tau} = \frac{1}{8\pi\epsilon} - \frac{1}{8\pi\epsilon} e^{-\frac{1}{2\epsilon}(\sigma^2+\tau^2)} - \frac{1}{8\pi\epsilon} e^{-\frac{1}{2\epsilon}\sigma^2} + \frac{1}{8\pi\epsilon} e^{-\frac{1}{2\epsilon}\tau^2}$$

contributing

$$\delta \log(\mathrm{Det}' P^\dagger P)^{1/2} = \frac{1}{2}\frac{1}{16}\,\delta\rho + \ldots \tag{3.37}$$

This part of the determinant, along with $(\mathrm{Det}\,\Delta)^{-1/2}$, is not sewn, and the corner anomaly is minus that from a single boson, so this piece cancels the corner anomaly from the $X^0$ determinant. For the $\sigma$ components the heat kernel is

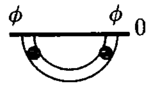
$$\mathcal{K}^{\sigma\sigma} = \frac{1}{8\pi\epsilon} + \frac{1}{8\pi\epsilon}e^{-\frac{1}{2\epsilon}(\sigma^2+\tau^2)} - \frac{1}{8\pi\epsilon}e^{-\frac{1}{2\epsilon}\sigma^2} - \frac{1}{8\pi\epsilon}e^{-\frac{1}{2\epsilon}\tau^2}.$$

The corner contribution is therefore

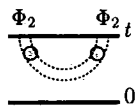
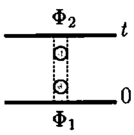
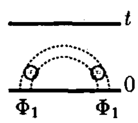
$$\delta\log(\mathrm{Det}\,'P^\dagger P)^{1/2} = -\frac{1}{2}\frac{1}{16}\,\delta\rho + \ldots \tag{3.38}$$

As we have seen the classical action for the $\sigma$ components involves $n^a\partial_a$, so we know from the Dirichlet case in (3.18) that this part of the determinant sews correctly without anomaly in the bulk.

We conclude that despite the extension of the string a Schrödinger representation makes sense for string field theory, and we can carry over our diagrammatic arguments and it follows that the vacuum wave functional and Schrödinger functional of string field theory are

$\Psi_0^{\mathrm{free}}[\Phi] = \exp\Big( -\frac{1}{4}$  $\Big)$ (3.39)


and

$\mathcal{S}^{\mathrm{free}}[\Phi_2, \Phi_1; t] = \exp\Big( -\frac{1}{2}$  $+$  $-\frac{1}{2}$  $\Big)$ (3.40)


respectively. The double line represents either the open or closed string propagator and the dotted line is the propagator with Dirichlet boundary conditions defined by a sum over images as in section (2.1). As is usual for free string field theory the centre of mass of $X^0$ acts as our time variable, though the $X^0$ oscillators are zero when the kernels in the above couple to the string field so our time is not extended and behaves as in particle theories.

The factorisation of the ghosts we have used may seem ad-hoc but in fact it follows from the gauge choice $P_{\hat{g}}^\dagger(g)^a = 0$ which is equivalent to the usual gauge fixing

$g_{ab} \propto \hat{g}_{ab}$. This, and the BRST symmetry are discussed in the final section of this chapter, where it is shown that BRST invariance corresponds to reparametrisation invariance of the boundary data attached to the propagator.

## 3.5 Light cone string field theory

Recall the light cone gauge string field theory [16],

$$S = \int \mathrm{d}\tau \int_{)}^{\infty} \mathrm{d}p_+ \int \mathcal{D}\mathbf{X}^{(24)}\ i\phi_{p+}^\dagger \partial_\tau \phi_{p+} - \frac{1}{2p_+}\phi_{p+}^\dagger h \phi_{p+} + S_{\text{int}}[\phi_{p+}^\dagger, \phi_{p+}] \tag{3.41}$$

where $\mathbf{X}$ is now the set of 24–dimensional transverse co-ordinates. In the Schrödinger representation we diagonalise the field operator at $\tau = 0$ and represent the algebra $[\phi, \phi^\dagger]$ acting on a basis of states $\langle\, \phi \,|$ by

$$\langle\, \phi \,| \hat{\phi}_{p+}[\mathbf{X}, 0] = \phi_{p+}[\mathbf{X}] \langle\, \phi \,|, \qquad \langle\, \phi \,| \hat{\phi}_{p+}^\dagger[\mathbf{X}, 0] = -\frac{\delta}{\delta \phi_{p+}[\mathbf{X}]} \langle\, \phi \,|. \tag{3.42}$$

The field dependence is made explicit by writing

$$\langle\, \phi \,| = \langle\, D \,| e^{-\int \phi \hat{\phi}^\dagger}.$$

The Dirichlet state $\langle\, D \,|$ is annihilated by $\hat{\phi}$. Taking the conjugate of these states we have a natural set of ket vectors which are eigenstates of $\hat{\phi}^\dagger$ rather than $\hat{\phi}$,

$$\begin{aligned} &\hat{\phi}_{p+}^\dagger[\mathbf{X}, 0] |\, \phi^\dagger \,\rangle = |\, \phi^\dagger \,\rangle \phi_{p+}^\dagger[\mathbf{X}], \qquad \hat{\phi}_{p+}[\mathbf{X}, 0] |\, \phi^\dagger \,\rangle = -\frac{\delta}{\delta \phi_{p+}^\dagger[\mathbf{X}]} |\, \phi^\dagger \,\rangle, \\ &|\, \phi^\dagger \,\rangle = e^{-\int \phi^\dagger \hat{\phi}} |\, N \,\rangle. \end{aligned} \tag{3.43}$$

The state $|\, N \,\rangle$ is annihilated by $\hat{\phi}^\dagger$. The matrix element of the time evolution operator between such states at times 0 and $T$ is

$$\begin{aligned} \langle\, \phi \,| e^{-i\hat{H}T} |\, \phi^\dagger \,\rangle &= \langle\, 0 \,| e^{-\int \hat{\phi}^\dagger \phi} e^{-i\hat{H}T} e^{-\int \phi^\dagger \hat{\phi}} |\, 0 \,\rangle \\ &= \int \mathcal{D}(\varphi, \varphi^\dagger) \exp\Bigg( i \int_0^T \mathrm{d}\tau\ L[\varphi, \varphi^\dagger] \\ &\qquad - \int \mathrm{d}p_+ \int \mathcal{D}\mathbf{X}\ \phi_{p+}^\dagger \varphi(T) + \phi_{p+} \varphi^\dagger(0) \Bigg) \Bigg|_{\varphi^\dagger = 0}^{\varphi = 0}. \end{aligned} \tag{3.44}$$

This has a Feynman expansion in terms of a propagator with mixed boundary conditions – in the Schrödinger functional only one boundary condition is specified for each of $\phi$ and $\phi^\dagger$ since the equations of motion are first order.

The free propagator $F$ for positive, Wick-rotated $\tau$ is [48], [49]

$$F(\delta\tau, \delta x^-, \delta\mathbf{x}) = \int_0^\infty \frac{\mathrm{d}p}{2p} e^{-ip\delta x^-} \left(\frac{p}{2\pi|\delta\tau|}\right)^{12} e^{-p\delta\mathbf{x}^2/2|\delta\tau|+|\delta\tau|/p}$$

$$\prod_{m=1} \left(1-e^{-2m|\delta\tau|/p}\right)^{-12} \exp\left(\frac{-2m}{\sinh(|\delta\tau|/p)}\left[(\mathbf{x}_m^{f\,2}+\mathbf{x}_m^{i\,2})\cosh(|\delta\tau|/p) - 2\mathbf{x}_m^f\cdot\mathbf{x}_m^i\right]\right)$$

with $x-$ being the Fourier transform of $p_+$. It is a simple matter to check the following Gaussian integral results

$$\int \mathfrak{D}\mathbf{X}\mathrm{d}x^- \; F(\tau_f, x^-, \mathbf{X}^f; \tau, x^-, \mathbf{X})\left(-2i\frac{\partial}{\partial x^-}\right)F(\tau, x^-, \mathbf{X}; \tau_i, x_i^-, \mathbf{X}_i)$$

$$= F(\tau_f, x_f^-, \mathbf{X}_f; \tau_i, x_i^-, \mathbf{X}_i) \quad \text{if } \tau_f > \tau > \tau +_i \tag{3.45}$$

$$= F_I(\tau_f, x_f^-, \mathbf{X}_f; \tau_i, x_i^-, \mathbf{X}_i) \quad \text{if } \tau_f, \tau_i > \tau \tag{3.46}$$

and $F_I$ is the propagator from one point to the reflection of another in the plane at time $\tau$. The insertion required is different to the conformal gauge case but should be expected since in the light cone gauge $-i\partial/\partial X^- = \pi_- = \dot{X}^+$. With this it is a simple matter to extend our arguments to the light cone gauge.

## 3.6 Gauge fixing and BRST invariance.

In this section we show that our ghost system follows from a choice of gauge equivalent to the usual choice $\sqrt{g}g^{ab} = \sqrt{\hat{g}}\hat{g}^{ab}(T)$, as mentioned earlier. We begin by showing this with Faddeev Popov gauge fixing and then investigate the BRST symmetry.

Our gauge fixing conditions are

$$\int \mathrm{d}^2\sigma\sqrt{g}\, g^{ab}\hat{h}_{ab} \equiv (\hat{h}_{ab}|g_{ab}) \quad = 0, \tag{3.47}$$

$$\hat{P}^\dagger\left(\frac{\sqrt{g}g^{rs}}{\sqrt{\hat{g}}}\right)^a \quad = 0. \tag{3.48}$$

These conditions are equivalent to the usual choice $\sqrt{g}g^{ab} = \sqrt{\hat{g}}\hat{g}^{ab}(T)$ as we now show. A variation of the metric can be decomposed into a Weyl scaling, a reparametrisation and a modular transformation as

$$\begin{aligned} \delta g_{ab} &= \delta\rho\, g_{ab} + \nabla_{(a}\delta\xi_{b)} + \delta T\, g_{ab,T}, & (3.49)\\ \Longrightarrow\ \delta(\sqrt{g}g^{ab}) &= -\sqrt{g}P(\delta\xi)^{ab} + \sqrt{g}\,\delta T\,\chi^{ab} & (3.50)\end{aligned}$$

where $\chi^{ab}$ is the traceless symmetric part of ${g^{ab}}_{,T}$. The first constraint above implies

$$\delta T(\hat{h}_{ab}|\hat{\chi}^{ab}) = 0 \implies \delta T = 0 \tag{3.51}$$

since the inner product is not zero. The second constraint gives

$$\hat{P}^\dagger\hat{P}\delta\xi^a = 0 \implies \delta\xi^a = 0 \tag{3.52}$$

by the boundary conditions on $\delta\xi^a$ (for the closed string we take $\delta\xi^a$ to be orthogonal to the CKV to get a good co-ordinate system). Inserting into (3.50) we have

$$\delta(\sqrt{\hat{g}}\hat{g}^{ab}) = 0 \implies \sqrt{g}g^{ab} = \sqrt{\hat{g}}\hat{g}^{ab}(T) \tag{3.53}$$

as claimed. We now compute the Faddeev-Popov determinant. Define the determinant $\Delta_{\mathrm{FP}}$ by

$$1 = \int \mathcal{D}\zeta \int \mathrm{d}T\ \Delta_{\mathrm{FP}}(g,T)\delta\big[(\hat{h}_{ab}\,|\,g^{ab})^\zeta\big]\prod_{a=1}^{2}\delta\left[\hat{P}^\dagger\left(\frac{\sqrt{g}g^{rs}}{\sqrt{\hat{g}}}\right)^a\right]^\zeta \tag{3.54}$$

where $\mathcal{D}\zeta$ is the measure on the diff $\times$Weyl group. This and $\Delta_{\mathrm{FP}}$ are invariant under the action of $\zeta$. Suppose that given $g$ the constraints have a solution $\zeta = (\rho, \xi^a)$ and $T$. Expanding about this solution gives

$$1 = \int \mathcal{D}(\delta\rho,\delta\xi)\int \mathrm{d}\delta T\ \Delta_{\mathrm{FP}}\ \delta\big[\delta T\,(\hat{h}_{ab}\,|\,\hat{\chi}^{ab})\big]\ \delta\big[\hat{P}^\dagger\hat{P}\delta\xi^a\big]. \tag{3.55}$$

We can integrate out the modular parameter and represent the delta functional by an integral over a vector Lagrange multiplier $\lambda^a$,

$$1 = \Delta_{\mathrm{FP}}(\hat{g})(\hat{h}\,|\,\hat{\chi}_{ab,T})^{-1}\int \mathcal{D}(\delta\rho,\delta\xi,\lambda)\exp\left(\frac{i}{2}\int \mathrm{d}^2\sigma\sqrt{\hat{g}}\ \lambda^a\hat{P}^\dagger\hat{P}\delta\xi^a\right). \tag{3.56}$$

The factor of one half multiplying the action is for later convenience. In the critical dimension the integral over the Liouville mode $\delta\rho$ contributes a volume (cancelling the corner anomalies if necessary). As usual we invert the expression for $\Delta_{\mathrm{FP}}$ by replacing bosonic with Grassmann variables. Following this and an integration by parts we have

$$\Delta_{FP}(\hat{g}) = (\hat{h} \,|\, \hat{\chi}_{ab,T}) \int \mathcal{D}(\gamma,\, c) \exp\left(\frac{1}{2}\int \mathrm{d}^2\sigma \sqrt{\hat{g}}\; \hat{P}(\gamma)^{ab}\hat{P}(c)_{ab}\right). \tag{3.57}$$

where $\gamma^a$, $c^a$ are Grassmann vectors obeying the Alvarez boundary conditions. We now wish to insert our expression for "1" into the Polyakov integral and carry out the integration over the metric. Since we known our gauge choice is equivalent to the usual choice we have two expressions for "1",

$$\begin{aligned} 1 &= \int \mathcal{D}\zeta \int \mathrm{d}T\, \Delta^0_{\mathrm{FP}}\delta(\sqrt{g}g - \sqrt{\hat{g}}\hat{g}) \\ &= \int \mathcal{D}\zeta \int \mathrm{d}T\; \Delta_{\mathrm{FP}}(g,T)\delta\left[(\hat{h}_{ab}\,|\, g^{ab})\right]\delta\left[\hat{P}^\dagger\left(\frac{\sqrt{g}g^{rs}}{\sqrt{\hat{g}}}\right)\right] \end{aligned} \tag{3.58}$$

where $\Delta^0_{FP}$ is the standard $b$-$c$ determinant. By changing variables $b^\perp = P\gamma$ in the expression for the FP determinants where $b^\perp$ is orthogonal to the zero mode of $\hat{P}^\dagger$ we find

$$\Delta^0_{\mathrm{FP}} = \Delta_{\mathrm{FP}}\mathrm{Det}'(\hat{P}^\dagger\hat{P})^{-1/2} = \Delta_{\mathrm{FP}}\int \mathcal{D}f\; e^{-\int \hat{P}f.\hat{P}f/2} \tag{3.59}$$

and we can carry out the integration over $g$ in our functional integral using the usual, single, delta function,

$$\int \mathcal{D}g \frac{1}{\text{Vol diff}\times\text{Weyl}} = \int \mathrm{d}T\, (\hat{h}_{ab}|\hat{\chi}^{ab}) \int \mathcal{D}(\gamma, c, f) e^{\int \hat{P}\gamma\cdot\hat{P}c/2 - \int \hat{P}f.\hat{P}f/2}. \tag{3.60}$$

We now turn to the BRST quantisation of the string. The BRST action for our constraints is, setting $\alpha' = 1/2\pi$,

$$\begin{aligned} S_{\mathrm{BRST}} = \frac{1}{2}\int \mathrm{d}^2\sigma \sqrt{g}g^{ab}\partial_a X \partial_b X &+ \frac{1}{2}\delta_Q\left[\int \mathrm{d}^2\sigma \sqrt{\hat{g}}\; \hat{P}^\dagger\left(\frac{\sqrt{g}}{\sqrt{\hat{g}}}g^{rs}\right)^a \hat{g}_{ab}\gamma^b\right] \\ &+ \frac{1}{2}\delta_Q\left[\Gamma\int \mathrm{d}^2\sigma\sqrt{g}g^{ab}\hat{h}_{ab}\right] \end{aligned} \tag{3.61}$$

where $\gamma^a(\sigma,\tau)$ and $\Gamma$ are antighosts. The nilpotent BRST transformations are

$$\begin{aligned} \delta_Q X &= c^a\partial_a, X \\ \delta_Q\sqrt{g}g^{ab} &= -\sqrt{g}P(c)^{ab}, \\ \delta_Q\gamma^a &= iB^a, \quad \delta_Q B^a = 0, \\ \delta_Q\Gamma &= iB, \quad \delta_Q B = 0. \end{aligned} \tag{3.62}$$

Inserting these transformations into the action above we find, after an integration by parts,

$$\begin{aligned}S_{\mathrm{BRST}} = \frac{1}{2}\int \mathrm{d}^2\sigma\sqrt{g}g^{ab}\partial_a X\partial_b X + \frac{1}{2}\int \mathrm{d}^2\sigma\sqrt{g}\ (\hat{P}(\gamma)_{ab} + \Gamma\hat{h}_{ab})P(c)^{ab} \\ + \frac{i}{2}\int \mathrm{d}^2\sigma\sqrt{g}\ g^{ab}(\hat{P}(B)_{ab} + B\hat{h}_{ab}).\end{aligned} \tag{3.63}$$

Integrating over the Lagrange multipliers in the final term imposes the gauge fixing constraints (3.47). To find the BRST transformations which result after this integration we should solve the equations of motion of $g^{ab}$. Notice that the first two terms in $S_{\mathrm{BRST}}$ have exactly the $g$ dependence of the standard Faddeev-Popov action,

$$\frac{1}{2}\int \mathrm{d}^2\sigma\sqrt{g}g^{ab}\partial_a X\partial_b X + \frac{1}{2}\int \mathrm{d}^2\sigma\sqrt{g}\ b_{ab}P(c)^{ab}$$

with $b_{ab}$ replaced by $\hat{P}\gamma + \hat{h}\Gamma$. The equations of motion for $g$ therefore imply

$$2T_{ab} = -i\hat{P}(B)_a - i\hat{h}_{ab}B = -\delta_Q(\hat{P}(\gamma)_a + \hat{h}_{ab}\Gamma) \tag{3.64}$$

where $T$ is the usual energy momentum tensor of the string with $b_{ab} = \hat{P}(\gamma)_{ab} + \Gamma\hat{h}_{ab}$. The variations of $\gamma$ and $\Gamma$ can be disentangled by multiplying both sides by $\hat{h}^{ab}$, since $\hat{P}(\gamma)$ is orthogonal to $\hat{h}$. However, after imposing the constraints any dependence on $\hat{h}_{ab}$ vanishes from the action, which becomes

$$S_{\mathrm{BRST}} = \frac{1}{2}\int \mathrm{d}^2\sigma\sqrt{\hat{g}}\hat{g}^{ab}\partial_a X\partial_b X + \frac{1}{2}\int \mathrm{d}^2\sigma\sqrt{\hat{g}}\ \hat{P}(\gamma)_{ab}\hat{P}(c)^{ab}. \tag{3.65}$$

The BRST transformations reduce to

$$\begin{aligned}\delta_Q X &= c^a\partial_a X \\ \delta_Q c^a &= c^b\partial_b c^a \\ \delta_Q\hat{P}(\gamma)_{ab} &= -2\hat{T}_{ab}.\end{aligned} \tag{3.66}$$

The energy momentum tensor is $\hat{T} = \hat{T}^X + \hat{T}^{\mathrm{gh}}$,

$$\begin{aligned}2\hat{T}^X_{ab} &= \partial_a X\partial_b X - \frac{1}{2}\hat{g}_{ab}\partial^r X\partial_r X, \\ 2\hat{T}^{\mathrm{gh}}_{ab} &= \hat{P}(\gamma)_{p(a}\hat{\nabla}_{b)}c^p + (\hat{\nabla}_p\hat{P}(\gamma)_{ab})c^p - \hat{g}_{ab}\hat{P}(\gamma)_{rs}\hat{\nabla}^r c^s.\end{aligned} \tag{3.67}$$

We have a non-local transformation for the antighost. Just as is found in the standard case the transformation for the antighost is not nilpotent,

$$\delta^2\hat{P}(\gamma)_{ab} = -\mathcal{L}\hat{P}(c)_{ab}$$

where $\mathcal{L}$ is the Lagrangian density in (3.65). On a manifold without boundary, or a manifold with boundary where the Alvarez boundary conditions hold, we recover nilpotency when the ghost $c^a$ is on shell, for then $P^\dagger Pc = 0 \iff P(c) = 0$.

Although the BRST transformation is now non-local, it has the natural interpretation of generating reparametrisations of the boundary as we now describe. Consider the string field propagator written as

$$G(\mathbf{X}_f; \mathbf{X}_i) = \int \mathcal{D}(X, \gamma, c)(\mathrm{Det}\, \hat{P}^\dagger \hat{P})^{-1/2} e^{-S_{\mathrm{BRST}} - S_J} \Big|_{X=X_i}^{X=X_f} \tag{3.68}$$

where $S_{\mathrm{BRST}}$ is as in (3.65) and $S_J$ is a source term which generates boundary values of the ghosts,

$$S_J = \frac{1}{2} \int \mathrm{d}^2\sigma \sqrt{\hat{g}}\ (\hat{P}^\dagger \hat{P}\gamma)_a c^a_{\mathrm{cl}} + \gamma^a_{\mathrm{cl}} (\hat{P}^\dagger \hat{P} c)_a.$$

In the above, $c^a_{\mathrm{cl}}$ obeys $\hat{P}^\dagger \hat{P} c^a_{\mathrm{cl}} = 0$ and equals the boundary values of the ghosts on the Dirichlet sections of the worldsheet,

$$c^a(\sigma, 0) = c^a_i(\sigma), \qquad c^a(\sigma, 1) = c^a_f(\sigma),$$

and similarly for $\gamma^a$. The integration variables obey the boundary conditions (3.26) and $c^T_{\mathrm{cl}} = \gamma^T_{\mathrm{cl}} \equiv 0$. After repeated integration by parts we can write the source term as

$$S_J = \int_{\mathrm{bhd}} \mathrm{d}\Sigma^b\ (\hat{P}\gamma)_{ab} c^a_{\mathrm{bhd}} + \gamma^a_{\mathrm{bhd}} (\hat{P}c)_{ab}. \tag{3.69}$$

The bulk action $S_{\mathrm{BRST}}$ is invariant for arbitrary boundary values of $X, c, \gamma$,

$$\delta_Q S_{\mathrm{BRST}} = -\int \mathrm{d}^2\sigma \frac{\partial}{\partial \sigma^a} \left( c^a \mathcal{L} \right) = 0$$

using the ghost boundary conditions $n^a c_a = 0$. The source term $S_J$ does not respect this symmetry, so we are led to expect a Ward identity resulting from a shift in integration variables corresponding to (3.66),

$$\langle\, \delta_Q S_J \,\rangle = 0 \tag{3.70}$$

where the expectation value is defined as

$$\langle \Omega \rangle := \int \mathcal{D}(X, \gamma, c)\ \Omega\ (\mathrm{Det}\, \hat{P}^\dagger \hat{P})^{-1/2} e^{-S_{\mathrm{BRST}} - S_J} \Big|_{X=X_i}^{X=X_f}$$

so the propagator itself is $\langle 1 \rangle$. To simplify the remaining presentation we now fix the gauge $\hat{g}_{ab} = \delta_{ab}$ and absorb dependency on the modular parameter into the co-ordinates, so $\sigma \in [0, \pi]$ and $\tau \in [0, T]$. Consider calculating the expectation value of the ghosts away from the boundary. Each $c^a$ ($\gamma^a$) is contracted with $\gamma^a$ ($c^a$) in the source term and brings down the classical field,

$$\langle\, c^a(\sigma,\tau)\,\rangle = c^a_{cl}(\sigma,\tau), \qquad \langle\, \gamma^a(\sigma,\tau)\,\rangle = \gamma^a_{cl}(\sigma,\tau). \tag{3.71}$$

As $\tau \to 0, 1$ this gives the boundary values of the ghosts. In general an expectation value separates into quantum and classical pieces. In addition to the usual short distance divergences the quantum pieces will contain finite, non-zero contributions from the image charges i.e. the corner anomaly contributes to the Ward identity. To illustrate, the contribution to (3.70) from the first term in (3.69), at $\tau = \epsilon > 0$ is

$$\langle \delta_Q \int \mathrm{d}\sigma\, (P\gamma)_{\sigma\tau} c^\sigma_{\mathrm{b}} \rangle = -\int \mathrm{d}\sigma \,\big\langle\, X'\dot{X} + (P\gamma)_{p(\sigma}\partial_{\tau)} c^p + (\partial_p (P\gamma)_{\sigma\tau}) c^p \,\big\rangle\, c^\sigma_{\mathrm{cl}}(\sigma,\epsilon). \tag{3.72}$$

The two boundaries contribute similarly so we will focus on $\tau = 0$, and let a subscript "b" denote boundary value. We will deal with the co-ordinate term first. The $X$ functional integrals are carried out by splitting $X$ into classical and quantum parts, $X = X_{\mathrm{cl}} + X_q$. The expectation value becomes

$$-\int \mathrm{d}\sigma\;\, c^\sigma_{\mathrm{cl}}(\sigma,\epsilon) X'_{\mathrm{cl}}(\sigma,\epsilon)\dot{X}_{\mathrm{cl}}(\sigma,\epsilon) + c^\sigma_{\mathrm{cl}}(\sigma,\epsilon)\big\langle\, X'_q(\sigma,\epsilon)\dot{X}_q(\sigma',\epsilon)\big\rangle.$$

To calculate the quantum contribution from the corners, we again go to the upper right quadrant with Dirichlet conditions at $\tau = 0$ and Neumann conditions at $\sigma = 0$. The Green's function, $F$, for $X_q$ on this geometry is given by the method of images,

$$F(\sigma,\tau;\sigma',\tau') = F_0(\sigma,\tau;\sigma',\tau') + F_0(\sigma,\tau;-\sigma',\tau') - F_0(\sigma,\tau;\sigma'\,-\tau') - F_0(\sigma,\tau;-\sigma',-\tau') \tag{3.73}$$

in terms of the free space Green's function

$$F_0(\sigma,\tau;\sigma'\tau') = -\frac{1}{4\pi}\log\big((\sigma-\sigma')^2 + (\tau-\tau')^2\big). \tag{3.74}$$

The contribution we are interested in comes from the final term in (3.73), as this is the only term which sees both reflections and hence the corner. Taking $\sigma = \sigma'$ and $\tau = \tau' = \epsilon$ this term contributes

$$-\frac{1}{4\pi}\frac{\sigma}{\sigma^2+\epsilon^2}\frac{\epsilon}{\sigma^2+\epsilon^2} \to -\frac{1}{4}\frac{\delta(\sigma)}{\sigma} \quad \text{as } \epsilon \to 0. \tag{3.75}$$

Since each corner contributes equally and independently we know that the expectation value on the strip will be given by

$$\langle X'_q(\sigma,0)\dot{X}_q(\sigma',0)\rangle = -\frac{1}{4}\frac{\delta(\sigma)}{\sigma} - \frac{1}{4}\frac{\delta(\sigma-\pi)}{\sigma-\pi} \tag{3.76}$$

per spacetime dimension. This leads to a quantum contribution to the Ward identity,

$$-\int_0^\pi d\sigma\; c^\sigma_{\mathrm{b}}(\sigma)\Big(-\frac{1}{4}\frac{\delta(\sigma)}{\sigma} - \frac{1}{4}\frac{\delta(\sigma-\pi)}{\sigma-\pi}\Big) = \frac{1}{4}\Big({c^\sigma}_{\mathrm{b}}'(0) + {c^\sigma}_{\mathrm{b}}'(\pi)\Big), \tag{3.77}$$

where we have applied L'Hôpital's rule since $c^\sigma(0) = c^\sigma(\pi) = 0$. As we take $\epsilon \to 0$ the total contribution from the co-ordinates to the Ward identity is

$$-\int d\sigma\, {c^\sigma}_{\mathrm{b}}(\sigma)X'_{\mathrm{b}}(\sigma)\dot{X}_{\mathrm{cl}}(\sigma,0) + \frac{1}{4}\Big({c^\sigma}_{\mathrm{b}}'(0) + c^{\sigma\prime}_{\mathrm{b}}(\pi)\Big). \tag{3.78}$$

The remaining classical field can be written as a derivative acting on the propagator $\langle 1\rangle$ with respect to the $X$ boundary data, so we have the operator expression

$$2\int d\sigma\, c^\sigma_{\mathrm{b}}(\sigma)X'_{\mathrm{b}}(\sigma)\frac{\delta}{\delta X_{\mathrm{b}}(\sigma)} + \frac{1}{4}\Big(c^{\sigma\prime}_{\mathrm{b}}(0) + c^{\sigma\prime}_{\mathrm{b}}(\pi)\Big) \tag{3.79}$$

per co-ordinate. This looks like a reparametrisation apart from the terms coming from the corners. We will return to these extra terms in Chapter 5.

All that remains is to calculate the ghost contributions from (3.69) to the Ward identity. Recall that we found the total contribution to the corner anomaly in $\mathrm{Det}\, P^\dagger P$ vanished using our ghosts, cancelling between the $\sigma$ and $\tau$ components. The classical ghost terms in (3.72) are (we have re-ordered some terms)

$$-\int d\sigma\;\; c^\sigma_{\mathrm{b}}(\sigma){\gamma^{\sigma\prime}}_{\mathrm{b}}(\sigma)\dot{c}^\sigma_{\mathrm{cl}}(\sigma,0) - c^\sigma_{\mathrm{b}}(\sigma){c^{\sigma\prime}}_{\mathrm{b}}(\sigma)\dot{\gamma}^\sigma_{\mathrm{cl}}(\sigma,0). \tag{3.80}$$

The quantum pieces are

$$\begin{aligned}
&-\int d\sigma\,\langle\, P\gamma_{p(\sigma}\partial_{\tau)}c^p + (\partial_p(P\gamma)_{\sigma\tau})c^p\,\rangle\, c^\sigma_{\mathrm{cl}}(\sigma,\epsilon)\\
&= -\int d\sigma\,\langle\, \gamma^{\sigma\prime}\dot{c}^\sigma + \gamma^{\tau\prime}\dot{c}^\tau + \dot{\gamma}^\sigma c^{\sigma\prime} + \dot{\gamma}^\tau c^{\tau\prime} + \dot{\gamma}'^\sigma c^\sigma + \dot{\gamma}'^\tau c^\tau\,\rangle\, c^\sigma_{\mathrm{cl}}(\sigma,\epsilon).
\end{aligned} \tag{3.81}$$

The Green's function for $P^\dagger P$ on the upper right quadrant is given by

$$F_{ab} = \begin{pmatrix} F_{\sigma\sigma} & 0 \\ 0 & F_{\tau\tau} \end{pmatrix} \tag{3.82}$$

which is constructed using the method of images,

$$\begin{aligned}
F_{\sigma\sigma}(\sigma,\tau;\sigma',\tau') &= \frac{1}{2}F_0(\sigma,\tau;\sigma',\tau') - \frac{1}{2}F_0(\sigma,\tau;-\sigma',\tau') \\
&\qquad - \frac{1}{2}F_0(\sigma,\tau;\sigma'-\tau') + \frac{1}{2}F_0(\sigma,\tau;-\sigma',-\tau'), \\
F_{\tau\tau}(\sigma,\tau;\sigma',\tau') &= \frac{1}{2}F_0(\sigma,\tau;\sigma',\tau') + \frac{1}{2}F_0(\sigma,\tau;-\sigma',\tau') \\
&\qquad - \frac{1}{2}F_0(\sigma,\tau;\sigma'-\tau') - \frac{1}{2}F_0(\sigma,\tau;-\sigma',-\tau').
\end{aligned} \tag{3.83}$$

Following the same argument as we used for the co-ordinates it is a simple matter to check that the corner contributions once again cancel between the $\sigma-$ and $\tau-$components.

Finally, the second term in (3.69) contributes only a classical piece,

$$\langle \delta_Q \int \mathrm{d}\sigma\, \gamma^\sigma_{\mathrm{b}}(Pc)_{\sigma\tau}\rangle = -\int \mathrm{d}\sigma\, \gamma^\sigma \langle \partial_\tau(c^\sigma c^{\sigma\prime} + c^\tau \dot{c}^\sigma)\rangle = -\int \mathrm{d}\sigma\, 2c^{\sigma\prime}_{\mathrm{b}}\gamma^\sigma_{\mathrm{b}}\dot{c}^\sigma_{\mathrm{cl}} + c^\sigma_{\mathrm{b}}\gamma^{\sigma\prime}_{\mathrm{b}}\dot{c}^\sigma_{\mathrm{cl}}, \tag{3.84}$$

since for the quantum pieces $\langle cc\rangle = 0$ as the ghosts anti commute. We can turn this into an operator acting on the propagator using

$$\dot{c}^\sigma_{\mathrm{cl}} = -\frac{\delta}{\delta\gamma^\sigma_{\mathrm{b}}}, \qquad \dot{\gamma}^\sigma_{\mathrm{cl}} = \frac{\delta}{\delta c^\sigma_{\mathrm{b}}}, \qquad \dot{X}_{\mathrm{cl}} = -2\frac{\delta}{\delta X_{\mathrm{b}}}, \tag{3.85}$$

where the factors come from the conventions in the action. Collecting all the terms we find the Ward identity

$$\left[\int_0^\pi \mathrm{d}\sigma\; c^\sigma_{\mathrm{b}}\mathbf{X}'_{\mathrm{b}}\frac{\delta}{\delta\mathbf{X}_{\mathrm{b}}} + \left(c^\sigma_{\mathrm{b}}\gamma^\sigma_{\mathrm{b}}\right)'\frac{\delta}{\delta\gamma^\sigma_{\mathrm{b}}} + \frac{1}{2}c^\sigma_{\mathrm{b}}c^{\sigma\prime}{}_{\mathrm{b}}\frac{\delta}{\delta c^\sigma_{\mathrm{b}}} + \frac{26}{8}\left(c^{\sigma\prime}(0) + c^{\sigma\prime}(\pi)\right)\right]G = 0. \tag{3.86}$$

This operator describes the transformation of 25 scalars, $\mathbf{X}$ ($X^0$ drops out since its tangential derivative is zero on the boundary) and the tangential component of a vector $\gamma^\sigma$, under a reparametrisation of the boundary generated by the ghost $c^\sigma$, and quantum corrections. The nonlocal BRST transformations correspond to local reparametrisations of the boundary as claimed. Note that demanding BRST invariance here does not put the string field on shell, as the reparametrisations are only a subset of those described by BRST in the usual formalism.

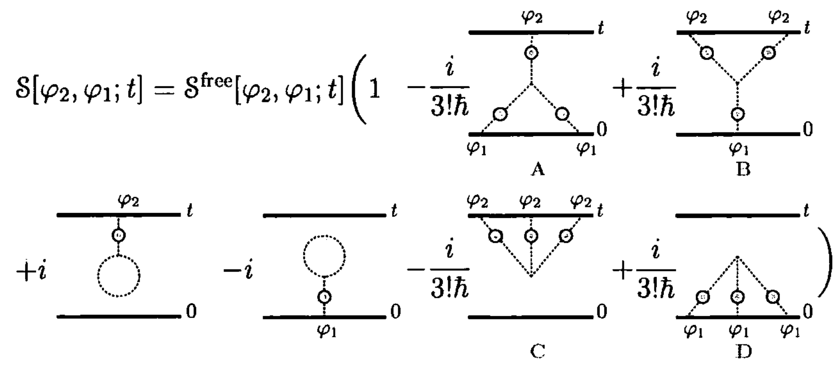
# Chapter 4

# Interacting field and string field theories

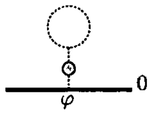
We begin by showing the equivalence of our diagrammatic methods and covariant methods in interacting scalar field theory. Since the string field interaction is, in various guises, cubic, we will consider here $\phi^3$ theory. We will then describe an alternative method of constructing field theory functionals which can be generalised to non-locally interacting field theories and can be used, as in [50], to construct the string field vacuum functional.

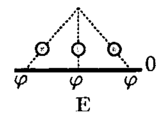
## 4.1 Interacting correlation functions

Using the functional integral definitions in Section (2.1) the first order vacuum wave functional and Schrödinger functional for $\phi^3$ theory are



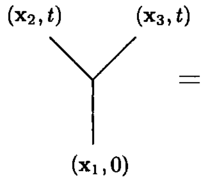

$$, \qquad (4.1)$$

$$\Psi_0[\varphi] = \Psi_0^{\text{free}}[\varphi]\Bigg(1 + i$$ 

$$- \frac{i}{3!\hbar}$$ 

$$\Bigg) \qquad (4.2)$$

The gray letters will be used to reference the diagrams. We will calculate the simplest non-trivial correlation function




$$= -i\lambda\hbar^2 \int_{-\infty}^{\infty} \mathrm{d}^D\mathbf{z}\mathrm{d}z \; G_0(\mathbf{x}_1, 0; \mathbf{z}, z) G_0(\mathbf{z}, z; \mathbf{x}_2, t) G_0(\mathbf{z}, z; \mathbf{x}_3, t) \qquad (4.3)$$

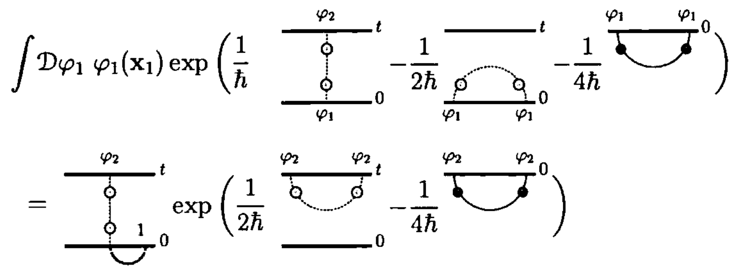
In the Schrödinger representation this is given by the functional integral

$$\int \mathcal{D}(\varphi_2, \varphi_1) \; \Psi_0[\varphi_2] \, \varphi_2(\mathbf{x}_3)\varphi_2(\mathbf{x}_2) \, \mathcal{S}[\varphi_2, \varphi_1; t] \, \varphi_1(\mathbf{x}_1)\Psi_0[\varphi_1]. \qquad (4.4)$$

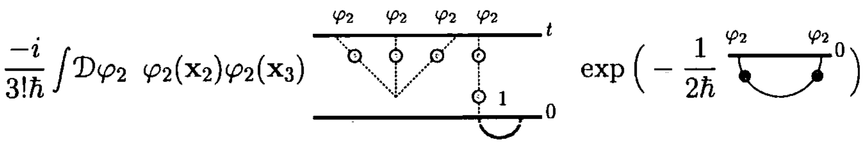
Excluding the loop diagrams, which can only contribute disconnected graphs and are of the wrong order in $\hbar$, we must include all of the diagrams in (4.1) and (4.2) to calculate (4.3), not just diagram B which pictorially matches the desired result. The reason is that, besides being of the correct order to contribute, even though the Dirichlet propagators may not end on a boundary, their boundary conditions mean they can still see it, and can contribute to the free space result.

Inserting expressions (4.1) and (4.2) into (4.4), carrying out the functional integrals and keeping only the order $\lambda$ terms, the diagrams A to E contribute the set of diagrams shown below (excluding loops or disconnected pieces generated by the integrations). To illustrate we will give the calculation for diagram C explicitly, and state the results for the remaining diagrams. Below, the numbers 1 to 3 indicate the spatial positions of the external legs and the thick gray line will be shorthand for $K^{-1}$ which we met in Chapter 2.

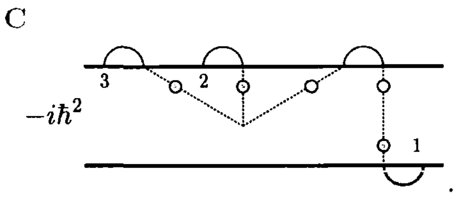
We now begin the computation of diagram C. In this case the $\varphi_1$ integral does not see the contribution from the interacting Schrödinger functional, and is Gaussian with the insertion $\varphi_1(\mathbf{x}_1)$,



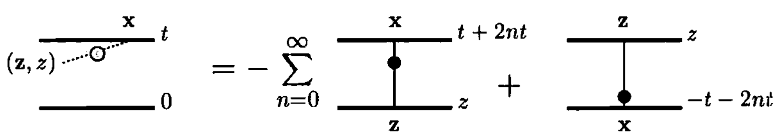
The $\varphi_2$ integral becomes



To obtain a connected graph we must contract $\varphi_2$, $\varphi_3$ with the 3 point function and the remaining 3 point function field must be contracted with the field attached to $\mathbf{x}_1$. There are 3! equivalent ways of doing this, so the result is

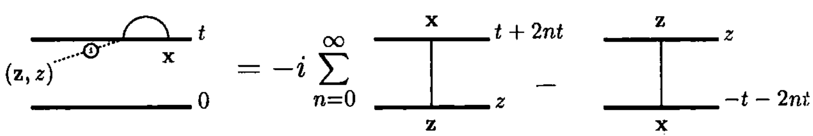


Using the method of images the propagator $G_D$ attached to one boundary is



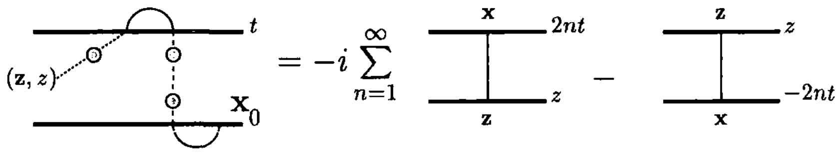
(4.5)

From which the gluing properties tell us that the two types of terms we encounter in diagram C are



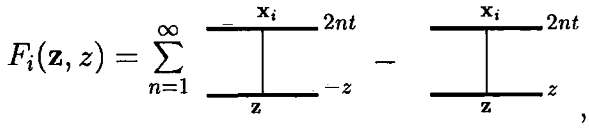
(4.6)

and

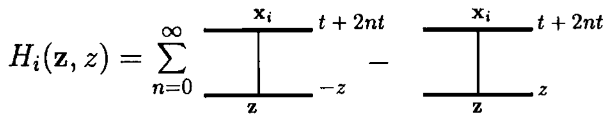



.   (4.7)

Defining the following sets of diagrams,




,   (4.8)



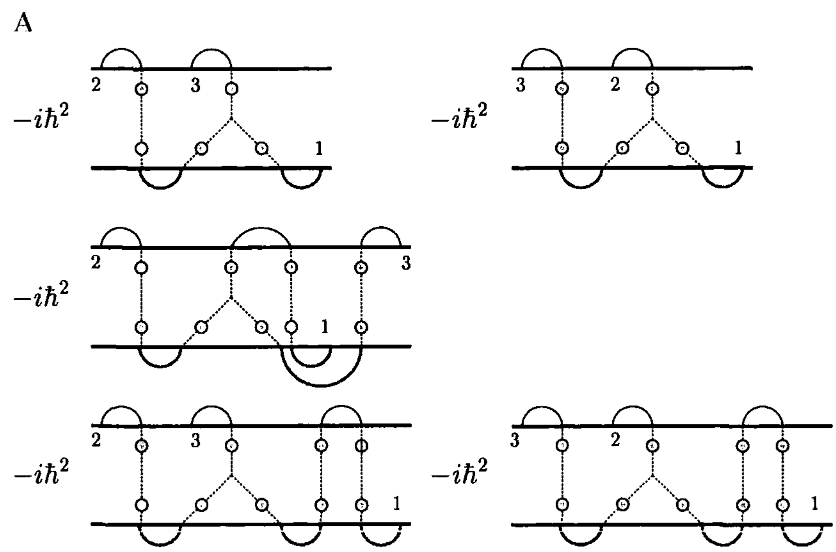

,   (4.9)

we can write the contribution from diagram C as

$$i\lambda\hbar^2\int_0^t \mathrm{d}z\int \mathrm{d}^D\mathbf{z}\; F_1(\mathbf{z},z)H_2(\mathbf{z},z)H_3(\mathbf{z},z).$$

The calculation of the remaining diagrams A to E are similar and the results are below. There is no content in the length of the $K^{-1}$ lines which varies in the diagrams only for clarity. $K^{-1}$ will appear on the lower boundary and the equal time propagator on the upper boundary if we perform the $\varphi_1$ integral first, and vice versa if we perform the $\varphi_2$ integral first.

Diagram A contributes



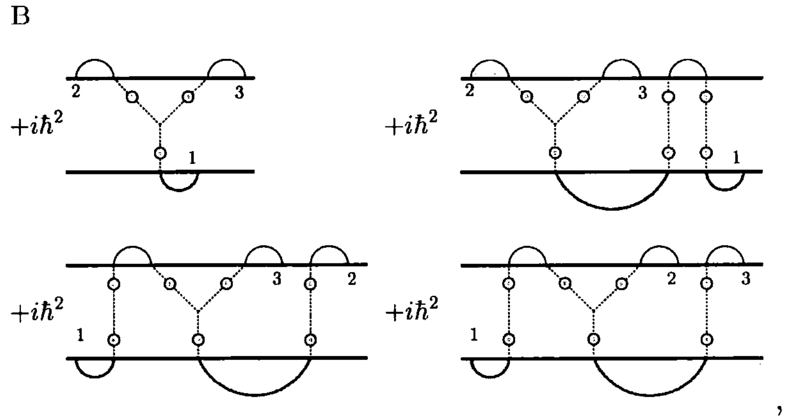
Written in terms of the sums $F_i(\mathbf{z},z)$ and $H_i(\mathbf{z},z)$ this is (the order of the terms is respective to that in the figures above)

$$i\lambda\hbar^2\int_0^t \mathrm{d}z\int \mathrm{d}^D\mathbf{z}\big(G_0(\mathbf{x}_1,0;\mathbf{z},z)-G_0(\mathbf{x}_1,2t;\mathbf{z},z)\big)H_2(\mathbf{z},z)\big(H_3(\mathbf{z},z)+G_0(\mathbf{z},t;\mathbf{x}_3,0)\big)$$
$$+(2\leftrightarrow 3)$$

$$+F_1(\mathbf{z},z)\big(H_2(\mathbf{z},z)+G_0(\mathbf{z},z;\mathbf{x}_2,t)\big)\big(H_3(\mathbf{z},z)+G_0(\mathbf{z},z;\mathbf{x}_3,t)\big)$$

$$+\big(F_1(\mathbf{z},z)+G_0(\mathbf{x}_1,2t;\mathbf{z},z)\big)\big(H_2(\mathbf{z},z)+G_0(\mathbf{z},z;\mathbf{x}_2,t)\big)H_3(\mathbf{z},z)+(2\leftrightarrow 3).$$

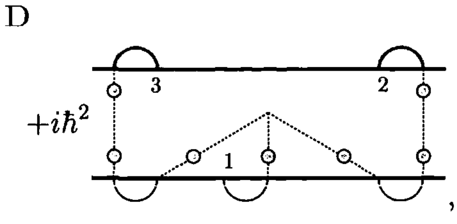
B contributes




equivalently,

$$-i\lambda\hbar^2\int_0^t \mathrm{d}z\int \mathrm{d}^D\mathbf{z}\;\big(G_0(\mathbf{x}_1,0;\mathbf{z},z)-G_0(\mathbf{x}_1,2t;\mathbf{z},z)\big)H_2(\mathbf{z},z)H_3(\mathbf{z},z)$$
$$+\big(F_1(\mathbf{z},z)+G_0(\mathbf{x}_1,2t;\mathbf{z},z)\big)H_2(\mathbf{z},z)H_3(\mathbf{z},z)$$
$$+F_1(\mathbf{z},z)\big(H_2(\mathbf{z},z)+G_0(\mathbf{z},z;\mathbf{x}_2,t\big)\big(H_2(\mathbf{z},z)+G_0(\mathbf{z},z;\mathbf{x}_2,t\big)H_3(\mathbf{z},z)$$
$$+(2\leftrightarrow 3).$$

Diagram D gives



which is

$$-i\lambda\hbar^2\int_0^t \mathrm{d}z\int \mathrm{d}^D\mathbf{z}\ \big(F_1(\mathbf{z},z)+G_0(\mathbf{x}_1,0;\mathbf{z},z)\big)\big(H_2(\mathbf{z},z)+G_0(\mathbf{z},z;\mathbf{x}_2,t)\big)$$

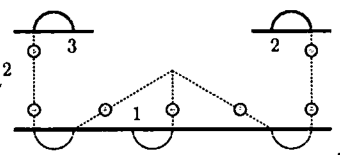
$$\times\ \big(H_3(\mathbf{z},z)+G_0(\mathbf{z},z;\mathbf{x}_3,t)\big).$$

Before calculating the diagrams coming from the interacting vacuum functional it is worthwhile adding the above terms up to find, with no calculation other than cancellations, that the total contribution from the interacting Schrödinger functional and the free vacuum is

$$-i\lambda\hbar^2\int_0^t \mathrm{d}z\int \mathrm{d}^D\mathbf{z}\ G_0(\mathbf{x}_1,0;\mathbf{z},z)G_0(\mathbf{z},z;\mathbf{x}_2,t)G_0(\mathbf{z},z;\mathbf{x}_3,t). \tag{4.10}$$

This is the correlation function we were looking for (with the correct coefficient) but the vertex is integrated in time only over the interval $[0,t]$. What remains must come from the vacuum. The contribution from the interacting vacuum functional $\Psi_0[\varphi_1]$, diagram E, is

E

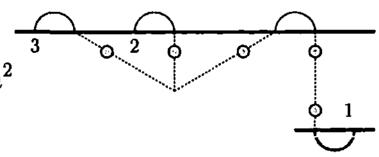
$-i\hbar^2$  .

The upper boundary (at time $t$) is broken as a reminder that only the outermost Dirichlet propagators vanish on both boundaries (i.e. come from the free field Schrödinger functional), whereas the inner three propagators vanish only at $x^0=0$ (come from the free field vacuum). This is

$$-i\lambda\hbar^2\int_{-\infty}^0 \mathrm{d}z\int \mathrm{d}^D\mathbf{z}\ G_0(\mathbf{x}_1,0;\mathbf{z},z)G_0(\mathbf{z},z;\mathbf{x}_2,t)G_0(\mathbf{z},z;\mathbf{x}_3,t) \tag{4.11}$$

There is a contribution from the vacuum $\Psi_0[\varphi_2]$, which we call term F,

F

$-i\hbar^2$ 

Again, only the rightmost propagator comes from the Schrödinger functional. This is

$$-i\lambda\hbar^2\int_{-\infty}^{0} \mathrm{d}z\int \mathrm{d}^D\mathbf{z}\ G_0(\mathbf{x}_1, z; \mathbf{z}, t)G_0(\mathbf{z}, z; \mathbf{x}_2, 0)G_0(\mathbf{z}, z; \mathbf{x}_3, 0). \tag{4.12}$$

Changing variable

$$z \to -z + t \tag{4.13}$$

in $F$ we have a total contribution of

$$\begin{aligned} -i\lambda\hbar^2\int_{-\infty}^{0} \mathrm{d}z\int \mathrm{d}^D\mathbf{z}\ G_0(\mathbf{x}_1, 0; \mathbf{z}, z)G_0(\mathbf{z}, z; \mathbf{x}_2, t)G_0(\mathbf{z}, z; \mathbf{x}_3, t) \\ - i\lambda\hbar^2\int_{t}^{\infty} \mathrm{d}z\int \mathrm{d}^D\mathbf{z}\ G_0(\mathbf{x}_1, 0; \mathbf{z}, z)G_0(\mathbf{z}, z; \mathbf{x}_2, t)G_0(\mathbf{z}, z; \mathbf{x}_3, t) \end{aligned} \tag{4.14}$$

This is again the integrand we want, but we are missing the integral over the region $[0, t]$. This missing piece is precisely what we found to be contributed by the Schrödinger functional in (4.10).

We have seen that for this simple scattering process the interacting vacuum wave functional gives almost the correct result, up to some 'small' (in the sense that the remaining term is integrated over only a finite time) missing piece, and that this is given by the Schrödinger functional.

## 4.2 Reconstruction of the vacuum wave functional

The aim of this section is to use diagrammatic methods to show that the interacting vacuum wave functional for $\phi^3$ theory can be constructed from the requirement that it must generate known results for vacuum expectation values (equal time correlation functions) through

$$\begin{aligned} \langle\, \phi(\mathbf{x}_1, 0) \cdots \phi(\mathbf{x}_n, 0)\,\rangle &\equiv \langle\, 0\,|\hat{\phi}(\mathbf{x}_1, 0) \ldots \hat{\phi}(\mathbf{x}_n, 0)|\,0\,\rangle \\ &= \int \mathcal{D}\phi\ \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n)\ |\Psi_0[\phi]|^2. \end{aligned} \tag{4.15}$$

The gluing property will play a central role. We will build the vacuum wave functional perturbatively by order in $\hbar$ and $\lambda$, using the known diagram expansion of

$n$–point functions (A closely related approach to investigating the diagram expansion of the vacuum wave functional for $\phi^4$ theory has been considered in [38]– this is essential the reverse of the calculation we present here).

The free theory vacuum wave functional has the form

$$\Psi_0[\phi] = \exp\left(-\frac{1}{2\hbar}\int \mathrm{d}^D(\mathbf{x},\mathbf{y})\,\phi(\mathbf{x})\,\Gamma_2^0(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\right) \tag{4.16}$$

for some function $\Gamma_2^0$ and must generate the free space propagator restricted to the boundary $t=0$ via

$$\hbar G_0(\mathbf{x},0,\mathbf{y},0) = \langle \phi(\mathbf{x},0)\phi(\mathbf{y},0)\,\rangle = \int \mathcal{D}\phi\,\phi(\mathbf{x})\phi(\mathbf{y})e^{-\int\phi\Gamma_2^0\phi/\hbar} = \hbar\big(2\,\Gamma_2^0(\mathbf{x},\mathbf{y})\big)^{-1} \tag{4.17}$$

and we recover the result (2.15).

If we consider interactions then the logarithm of the vacuum wave functional has an expansion not only in powers of $\lambda$ but in $\hbar$ also. We expand the logarithm as

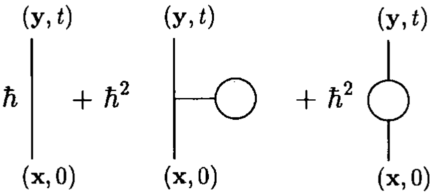
$$\log\Psi_0[\phi] = \frac{1}{\hbar}\sum_{n=1}^{\infty}\frac{i^n}{n!}\int \mathrm{d}^D(\mathbf{x}_1\ldots\mathbf{x}_n)\;\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\,\Gamma_n(\mathbf{x}_1,\ldots,\mathbf{x}_n) \tag{4.18}$$

where each of the $\Gamma_n$ has an expansion in powers of $\hbar$,

$$\Gamma_n = \Gamma_n^0 + \Gamma_n^{\hbar} + \Gamma_n^{\hbar^2} + \cdots$$

with their $\hbar$ dependence given by the superscript. There can be no correction to $\Gamma_2^0$ (defined in (4.16)) from the addition of interactions since this would give a leading order correction to the two point function, but we know that the first corrections are of order $\lambda^2\hbar^2$.

$\langle\phi(\mathbf{y},t)\phi(\mathbf{x},0)\rangle = \hbar$



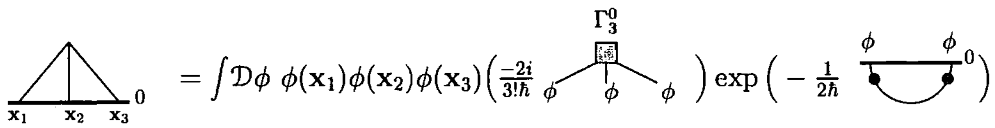

(4.19)

Keeping only $\Gamma_2^0$ in the exponent and expanding the other contributions to $\log\Psi_0[\phi]$, call this $\log\overline{\Psi}[\phi]$, we can write vacuum expectation values as

$$\langle\,\phi(\mathbf{x}_1,0)\ldots\phi(\mathbf{x}_n,0)\,\rangle = \int\mathcal{D}\phi\;e^{-\int\phi\Gamma_2^0\phi/\hbar}\sum_m\frac{(2\log\overline{\Psi}[\phi])^m}{m!}\,\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n), \tag{4.20}$$

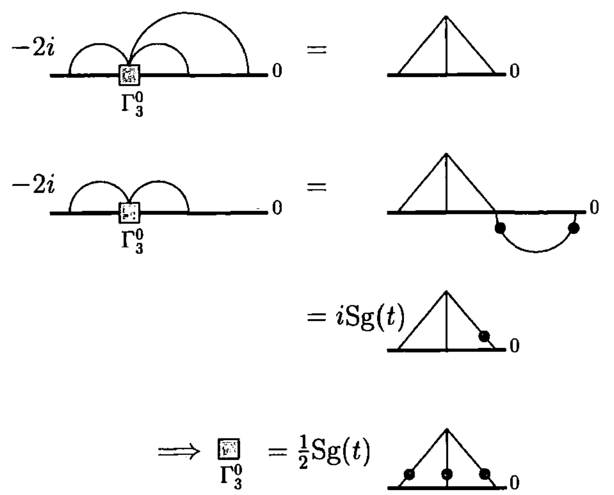
so in general we have to contract $\phi(\mathbf{x}_1)\ldots\phi(\mathbf{x}_n)$ with the $\phi$ in the diagrams contributing to $\log\overline{\Psi}$ using the inverse of $2\Gamma_2^0$ which is the equal time propagator in free space.

To identify the first interaction vertex $\Gamma_3^0$ consider the (connected) three point function




(4.21)

The box represents the kernel $\Gamma_0^3$, which has three arguments, see (4.18). When we perform the integral the three external fields are contracted with the three fields attached to $\Gamma_3^0$ using the equal time propagator, which must give us the usual three point function restricted to the boundary. This is represented by the first line in the equation below.



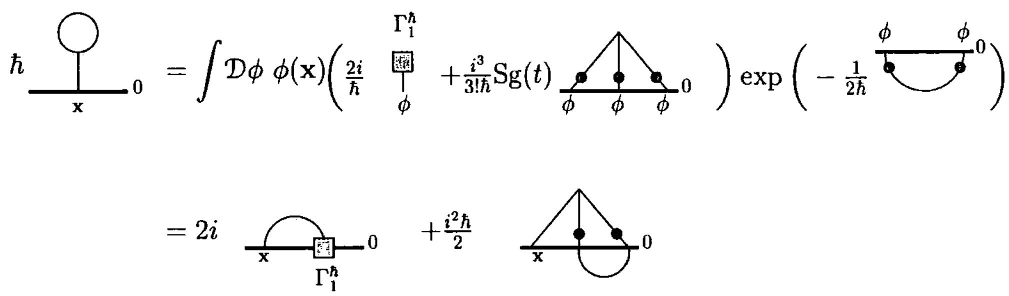

(4.22)

In the second line we invert one of propagators attached to $\Gamma_3^0$ using $2\Gamma_2^0$ to give the third line. The factor $\mathrm{Sg}(t)$, the sign of the vertex position $t$, emerges here because the gluing rules are dependent on the positions of all the endpoints of the propagators we are gluing. In the final line we have repeated these steps to remove the remaining two equal time propagators from $\Gamma_3^0$. A sign remains because the vertex is cubic. The signs do not appear in the $\phi^4$ calculations in [38] and [51] because of the even power of interacting fields. We trust the notation is not too confusing, the vertex is integrated over all space with the factor $\mathrm{Sg}(t)$ included in the integrand[1]. Keeping track of the combinatoric factors the explicit expression for the vertex is

$$\Gamma_3^0(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = -\lambda \int \mathrm{d}t \int \left[\prod_{j=1}^{3} \frac{\mathrm{d}^D\mathbf{k}_j}{(2\pi)^D} e^{-i\mathbf{k}_j.\mathbf{x}_j}\right](2\pi)^D \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) \times e^{-i(E(\mathbf{k})_1+E(\mathbf{k})_2+E(\mathbf{k})_3)|t|}.$$

All the tree level contributions $\Gamma_n^0$ can be similarly derived from considering the tree level $n$-point function at equal time, as we have done for $n=2,3$ above.

Using $\Gamma_3^0$ we can determine the tadpole $\Gamma_1^\hbar$. The tadpole graph is of order $\lambda\hbar$, so the only terms contributing are $\Gamma_3^0$ and $\Gamma_1^\hbar$. Expanding to first order in $\lambda$ we calculate

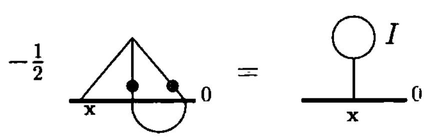



(4.23)

To make sense of the final diagram we appeal to the gluing property with appropriate conditions $t_1, t_2 \lessgtr 0$,

$$\int \mathrm{d}^D(\mathbf{x},\mathbf{y}) \left(2\frac{\partial}{\partial t}G_0(\mathbf{x},x_1;\mathbf{y},t)\right) G_0(\mathbf{y},0;\mathbf{z},0)\left(2\frac{\partial}{\partial t'}G_0(\mathbf{z},t';\mathbf{x}_2,t_2)\right)\Bigg|_{t'=t=0} = -G_I(\mathbf{x}_1,x_1;\mathbf{x}_2,t_2) \qquad (4.24)$$

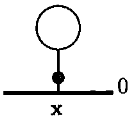
so the effect of gluing two free propagators with the inverse of $2\Gamma_0^2$ on the boundary is to produce the image propagator. The external leg becomes a free propagators and the internal lines become an image propagator loop, denoted by an $I$,

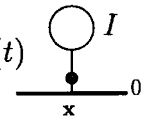



(4.25)

[1] In fact, since the derivatives on the propagators induce such signs as well, the signs do not appear explicitly, as can be seen in the given example.

The factor of $1/2$ on the left is the symmetry factor implicit in the diagram on the right. Loops of image and free propagators are unequal, so this term must be removed by $\Gamma_1^\hbar$. Inverting the equal time propagators we are led to

$\Gamma_1^\hbar = \frac{\hbar}{2}\mathrm{Sg}(t)$ 

$-\frac{\hbar}{2}\mathrm{Sg}(t)$ 

. (4.26)

Let us now compare this with (4.2). The Feynman diagrams there are constructed from $G_{d(0)}$ and the interaction vertices are integrated over the half space $t < 0$. The first order three field kernel is

$$-i\lambda \int_{-\infty}^{0} \mathrm{d}t \int \mathrm{d}^D\mathbf{y}\; G_{d(0)}(t,\mathbf{y};\dot{0},\mathbf{x}_1)G_{d(0)}(t,\mathbf{y};\dot{0},\mathbf{x}_2)G_{d(0)}(t,\mathbf{y};\dot{0},\mathbf{x}_3). \tag{4.27}$$

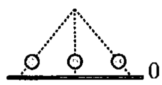
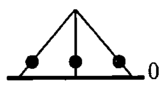
The derivative of $G_{d(0)}$ on the boundary obeys a Neumann boundary condition, so $G_{d(0)}$ can be replaced by $2G_0$. Now write this expression as two of the above terms each multiplied by a half, and in one term change variables $t \to -t$. The kernel becomes

$$\begin{aligned}
&-\frac{i\lambda}{2}\int_{-\infty}^{0} \mathrm{d}t \int \mathrm{d}^D\mathbf{y}\; 2G_0(t,\mathbf{y};\dot{0},\mathbf{x}_1).2G_0(t,\mathbf{y};\dot{0},\mathbf{x}_2).2G_0(t,\mathbf{y};\dot{0},\mathbf{x}_3)\\
&+\frac{i\lambda}{2}\int_{0}^{\infty} \mathrm{d}t \int \mathrm{d}^D\mathbf{y}\; 2G_0(t,\mathbf{y};\dot{0},\mathbf{x}_1).2G_0(t,\mathbf{y};\dot{0},\mathbf{x}_2).2G_0(t,\mathbf{y};\dot{0},\mathbf{x}_3).
\end{aligned} \tag{4.28}$$

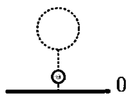
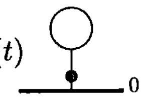
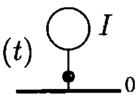
Finally we absorb the factors of 2 into the time derivative, and let a bullet • indicate $-2\partial/\partial t$, so that the kernel is

$$\begin{aligned}
&\frac{i\lambda}{2}\int_{-\infty}^{0} \mathrm{d}t \int \mathrm{d}^D\mathbf{y}\; G_0(t,\mathbf{y};\overset{\bullet}{0},\mathbf{x}_1)G_0(t,\mathbf{y};\overset{\bullet}{0},\mathbf{x}_2)G_0(t,\mathbf{y};\overset{\bullet}{0},\mathbf{x}_3)\\
&-\frac{i\lambda}{2}\int_{-\infty}^{0} \mathrm{d}t \int \mathrm{d}^D\mathbf{y}\; G_0(t,\mathbf{y};\overset{\bullet}{0},\mathbf{x}_1)G_0(t,\mathbf{y};\overset{\bullet}{0},\mathbf{x}_2)G_0(t,\mathbf{y};\overset{\bullet}{0},\mathbf{x}_3)\\
&= -\frac{i\lambda}{2}\int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}^D\mathbf{y}\; \mathrm{Sg}(t)G_0(t,\mathbf{y};\overset{\bullet}{0},\mathbf{x}_1)G_0(t,\mathbf{y};\overset{\bullet}{0},\mathbf{x}_2)G_0(t,\mathbf{y};\overset{\bullet}{0},\mathbf{x}_3)\\
&\equiv \Gamma_3^0.
\end{aligned} \tag{4.29}$$

So the Dirichlet Feynman diagram is equivalent to our construction using vacuum expectation values. This is more clearly represented by

 $= \frac{1}{2}\mathrm{Sg}(t)$  (4.30)

in agreement with our result (4.22). Similarly the tadpole in the vacuum wave functional obeys

 $= \frac{1}{2}\mathrm{Sg}(t)$  $-\frac{1}{2}\mathrm{Sg}(t)$  (4.31)

in agreement with (4.26).

## 4.3 Non-local interactions

The interaction Hamiltonian of local $\phi^3$ theory is

$$\int \mathrm{d}^D\mathbf{x}\ \frac{\lambda}{3!}\phi(\mathbf{x},t)^3. \tag{4.32}$$

Consider now the case when the interaction is local in time, but non-local in space so that the interaction Hamiltonian can be written in terms of some kernel $W$,

$$\lambda \int \prod_{j=1}^{3} \mathrm{d}^D\mathbf{x}_j\ W(\mathbf{x};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\phi(\mathbf{x}_1,t)\phi(\mathbf{x}_2,t)\phi(\mathbf{x}_3,t). \tag{4.33}$$

The calculations in the previous sections treated the spatial co-ordinates on an equal footing as standard covariant methods, only the time direction is singled out for special attention. As such, the calculation can be repeated for an arbitrary spatially non-local interaction.

Given this and our gluing property for string fields we may postulate any string field theory interaction which is non-local in the spatial co-ordinates and ghosts, but local in time, and where the fields depend only on $t$ and the set $\mathbf{B}$ defined in Chapter 3. For example we could postulate an interaction very like that in the light-cone gauge [16], with worldsheet $\tau$ replaced by spacetime $t$,

$$\int_{-\infty}^{\infty} \mathrm{d}t \int \mathcal{D}(\mathbf{B}, \mathbf{B}_1, \mathbf{B}_2) \Phi[\mathbf{B}(\sigma), t] \Phi[\mathbf{B}_1(\sigma_1), t] \Phi[\mathbf{B}_2(\sigma_2), t] \\ \prod_{0 \leq \sigma \leq \pi/2} \delta(\mathbf{B}(\sigma) - \mathbf{B}_1(\sigma_1)) \prod_{\pi/2 \leq \sigma \leq \pi} \delta(\mathbf{B}(\sigma) - \mathbf{B}_2(\sigma_2)) \tag{4.34}$$

where all three fields interact at the same time and the parameters along the string are

$$\begin{aligned} \sigma_1 &= 2\sigma, \quad 0 \leq \sigma \leq \pi/2 \\ \sigma_2 &= 2\sigma - \pi, \quad \pi/2 \leq \sigma \leq \pi \end{aligned} \tag{4.35}$$

so that each string has parameter domain $0 \leq \sigma \leq \pi$. If the free Hamiltonian was the inverse of $G(\mathbf{B}_2, t_2; \mathbf{B}_1, t_1)$ we could repeat any quantum field theory argument in this theory. However, this approach removes the $X^0$ oscillators from the theory from the outset, and it is unclear how to justify this nor how amplitudes in this theory relate to known results. Since non-locality in one direction does not spoil the UV properties of string interactions (as in the light cone) [17] this may be a worthwhile avenue to explore, but here we will stick to conventional string field interactions.

In the next section we will construct the string field theory vacuum. To ready us we describe the construction of the vacuum wave functional for a quantum field theory with an interaction non-local in time. If we try to apply the functional integral prescription given earlier we do not obtain the correct expression for the vacuum wave functional. The problem is that we are trying to write the vacuum in terms of field histories in the half space $t < 0$, but the non-local Hamiltonian depends on the field at all times. For similar reasons, the Schrödinger functional integral description also fails. We will instead use the reconstructive arguments of the previous section.

## Tree level, first order

The free theory vacuum wave functional is unchanged. Let the non-local cubic Hamiltonian be

$$\lambda \int_{-\infty}^{\infty} \prod_{j=1}^{3} \mathrm{d}t_j \;\; W(t; t_1, t_2, t_3) \phi(t_1) \phi(t_2) \phi(t_3). \tag{4.36}$$

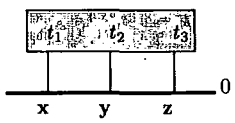
The vertex may be local or non-local in space, so we will suppress the dependence on the spatial co-ordinates and will continue to do so whenever possible. Accordingly, we will use the notation

$$G_{\mathbf{x}}(0;t_j) := G_0(\mathbf{x},0;\mathbf{x}_j,t_j) \tag{4.37}$$

for the free space propagator. A covariant field theory calculation implies that the lowest order contribution to the three field vacuum expectation value is

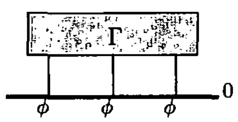
$$\begin{aligned}\langle\, 0\,|\phi(\mathbf{x},0)\phi(\mathbf{y},0)\phi(\mathbf{z},0)|\,0\,\rangle = -i\lambda\hbar^2 \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \prod_{j=1}^{3} \mathrm{d}t_j \;\; W(t;t_1,t_2,t_3)\\ \times \sum_{\text{perms of } t_j} G_{\mathbf{x}}(0;t_1)G_{\mathbf{y}}(0;t_2)G_{\mathbf{z}}(0;t_3)\end{aligned} \tag{4.38}$$

We get a sum over permutations since any external leg can be contracted to any argument of the vertex kernel but the ways of contracting them will not, in general, be equivalent. Alternatively, we can abandon the explicit kernel $W$ and think of this as some Feynman diagram which ties together three legs ending at $t = 0$, in some way:

$\sum_{\text{perms of } t_i}$ 

. (4.39)

Introducing an unknown diagram $\Gamma$ into the vacuum wave functional,

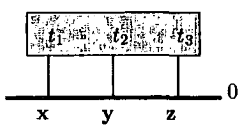
$\Psi_0[\phi] = \Psi_0[\phi]^{\text{free}}\Big(1+$ 

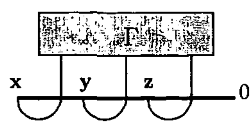
$\Big)$ . (4.40)

we require the above expectation value to be given by

$$\langle\, 0\,|\phi(\mathbf{x},0)\phi(\mathbf{y},0)\phi(\mathbf{z},0)|\,0\,\rangle = \int \mathcal{D}\phi \;\; \phi(\mathbf{x})\phi(\mathbf{y})\phi(\mathbf{z})\Psi_0^2[\phi],$$

equivalently,

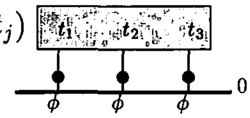
$\sum_{\text{perms of } t_i}$ 

$= 2\hbar^3 \sum_{\text{perms}}$ 

. (4.41)

To identify $\Gamma$ we can invert these equal time propagators as before. The result of applying this gluing to the left hand diagram depends on the signs of the $t_j$, and tells us that our blob $\Gamma$ is

$$\frac{1}{2\hbar^3}\prod_{j=1}^{3} i\mathrm{sign}(t_j)$$




. (4.42)

In terms of the interaction kernel $W$ this means

$$\Psi_0[\phi] = \Psi_0^{\mathrm{free}}[\phi]\left(1 - \frac{i\lambda}{2\hbar}\int_{-\infty}^{\infty} \mathrm{d}t \prod_{j=1}^{3}\mathrm{d}t_j \;\; W(t;t_1,t_2,t_3)\prod_{j=1}^{3} i\mathrm{sign}(t_j)\;\; \phi\cdot G(\overset{\bullet}{0};t_j)\right) \quad (4.43)$$

Notice the bullet over the 0 in the final factor, this is the time derivative with factor $-2$. We have used the shorthand

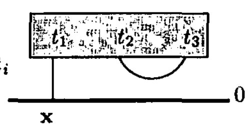
$$\phi\cdot G(\overset{\bullet}{0};t_j) := \int \mathrm{d}^D\mathbf{y} \;\; \phi(\mathbf{y})\, G_0(\mathbf{y},\overset{\bullet}{0};\mathbf{x}_j,t_j). \quad (4.44)$$

This is analogous to the vacuum in the local cubic theory, except that we now have a product of signs of the arguments $t_i$ of the interaction kernel $W$. These signs are missing if we try to use the functional integral definition of the vacuum.

## One loop, first order

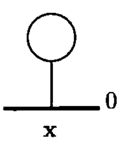
This term and that which we constructed above are the sole contributors to the tadpole graph,

$$\langle\, 0\,|\phi(\mathbf{x},0)|\,0\,\rangle = \sum_{\text{perms of } t_i}$$



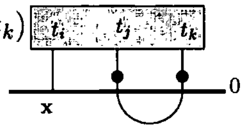

, (4.45)

which is just the generalisation of



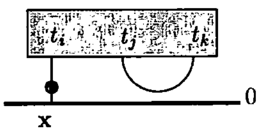

(4.46)

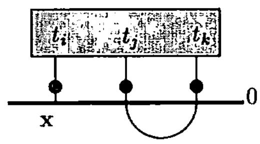
in local cubic theory. Calculating this expectation value with the vacuum wave functional we constructed above gives

$$\sum_{\text{perms of } t_1,t_2,t_3} i\mathrm{sign}(t_j).i\mathrm{sign}(t_k)$$



(4.47)

so we include in the vacuum wave functional a term which cancels this to give the correct result. Introducing a new term, carrying out the functional integral and inverting the external equal time propagators as in the previous subsection the first order one loop term in the vacuum wave functional is

$$\sum_{\text{perms of } t_j} i\mathrm{sign}(t_i)$$


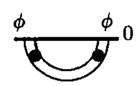
$$- \sum_{\text{perms of } t_j} \prod_{j=1}^{3} i\mathrm{sign}(t_j)$$



(4.48)

This is very similar to the result we found for local theories.

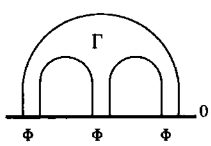
## 4.4 The vacuum of string field theory

Let us repeat the above arguments to construct the string field vacuum. As already mentioned, the free field vacuum is

$$\Psi_0^{\text{free}}[\Phi] = \exp\Bigg( -\tfrac{1}{4}$$


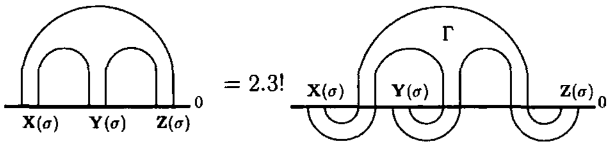
$$\Bigg) . \qquad (4.49)$$

We propose the lowest order expansion

$$\Psi_0[\Phi] = \Psi_0^{\text{free}}[\Phi]\Bigg(1+$$


$$\Bigg) . \qquad (4.50)$$

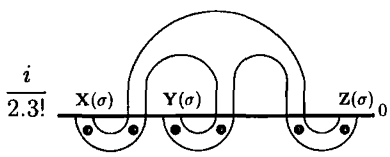
We would then try to recover the three field vacuum expectation value as we did before we find the relation




(4.51)

The diagram on the left is the three field vacuum expectation value, constructed with some three string vertex, or via the Polyakov integral [2] on a manifold with the topology of a disk with marked sections on the boundary (for clarity only we will assume that this three point function is symmetric under permutations of the data on its legs).

As before we invert the equal time propagators on the right hand side of the above equation, and the left hand side becomes the kernel we are trying to identify. The first order cubic term in the vacuum wave functional is therefore




(4.52)

Our gluing rules give us the result of joining propagators when each end lies on a constant time surface, they do not apply when one end lies on a time $X^0(\sigma)$ for non-trivial sigma dependence, nor how to attach more general surfaces together. For this we need Carlip's method, which sheds some light on what the above kernel (4.52) is.

In [47] it was shown that to correctly sew the moduli spaces of two worldsheets the inverse propagator (first quantised Hamiltonian) should be attached to one of the boundaries being sewn. This removes a redundant length parameter when we integrate over all shared field arguments on the boundaries being sewn which would otherwise give a divergent factor.

Let $T[t'_1, t'_2, t'_3]$ is the three string amplitude with external legs ending at curves $\mathbf{X} = \mathbf{X}_i(\sigma)$ and $X^0 = t'_i$. Using the simple identity

$$f(0) = \int \mathrm{d}s\ f(s)\delta(s) = \int \mathrm{d}s\mathrm{d}q\ f(s)G^{-1}(s,q)G(q,0). \qquad (4.53)$$

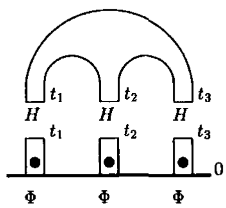
we can write the three field expectation value as

$$T[0,0,0] = \int \prod_{j=1}^{3} \mathrm{d}t'_j \mathrm{d}t_j\ T[t'_1, t'_2, t'_3] \prod_{i=1}^{3} G^{-1}(t'_i, t_i)G(t_i, 0) \qquad (4.54)$$

Now when we attach the inverse of the equal time propagator the gluing rules imply that the vacuum wave functional is

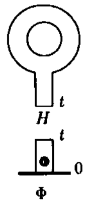
$$\Psi_0[\Phi] = \Psi_0^{\text{free}}[\Phi]\left(1 + \int \prod_{j=1}^{3} \mathrm{d}t'_j \mathrm{d}t_j \; T[t'_1, t'_2, t'_3] \prod_{i=1}^{3} G^{-1}(t'_i, t_i) \prod_{k=1}^{3} i\mathrm{sign}(t_k)\Phi \cdot G(\mathring{0}; t_k)\right). \tag{4.55}$$

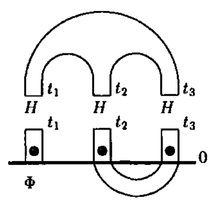
In our case the inverse propagator is that part of the first quantised string theory Hamiltonian which depends on $\mathbf{X}$ and $t$ on the boundary of our worldsheet. We can represent the vacuum functional as

$$\Psi_0[\Phi] = \Psi_0^{\text{free}}[\Phi]\left(1 + \tfrac{1}{2.3!}\prod_{j=1}^{3} i\mathrm{sign}(t_j)\right.$$


$$\left.\vphantom{\prod}\right) \tag{4.56}$$

where we have written $H$ in place of the inverse propagators for clarity. As before, the first order one loop term in the vacuum wave functional follows, and is

$\frac{i}{2}\mathrm{sign}(t)$


$-\frac{i}{2}\prod_{j=1}^{3} i\mathrm{sign}(t_j)$


. (4.57)

We have described, in analogy with quantum field theory, how to construct the string field theory vacuum wave functional. Note that we have not given the explicit form of the three string vertex. That choice is arbitrary, the required kernels could even, theoretically, be computed with the Polyakov integral on a the relevant manifold. Our construction applies to both open and closed strings.

Although it is a lengthier task, it is not especially more difficult to reconstruct the Schrödinger functional of quantum field theory using a similar approach, and the generalisation to string field theory will follow. We would propose the first order expansion

$$\mathcal{S}[\phi_2, \phi_1, t] = \mathcal{S}^{\text{free}}[\phi_2, \phi_1, t]\left(1 + \phi_2\phi_2\phi_2\mathcal{S}^{(3,0)} + \phi_2\phi_2\mathcal{S}^{(2,1)}\phi_1 + \phi_2\mathcal{S}^{(1,2)}\phi_1\phi_1 + \mathcal{S}^{(0,3)}\phi_1\phi_1\phi_1\right) \tag{4.58}$$

which includes four unknown kernels $\mathcal{S}^{(i,j)}$. There are four possible three field amplitudes which must be reproduced by a functional integration with one instance of the Schrödinger functional,

$$\langle\, 0\,|\phi(\mathbf{x},t)\phi(\mathbf{y},t)\phi(\mathbf{z},t)|\,0\,\rangle \qquad \langle\, 0\,|\phi(\mathbf{x},t)\phi(\mathbf{y},t)\phi(\mathbf{z},0)|\,0\,\rangle$$
$$\langle\, 0\,|\phi(\mathbf{x},t)\phi(\mathbf{y},0)\phi(\mathbf{z},0)|\,0\,\rangle \qquad \langle\, 0\,|\phi(\mathbf{x},0)\phi(\mathbf{y},0)\phi(\mathbf{z},0)|\,0\,\rangle$$

giving us four simultaneous functional equations with which to determine the kernels $\mathcal{S}^{(i,j)}$.

# Chapter 5

# The field momentum representation and T-duality

In this chapter we will describe an alternative representation of the state space of scalar quantum field theory which when generalised to string field theory reveals the effect of T-duality on the string field Schrödinger functional. This material expands on the discussion in [52]. Finally we present our conclusions.

## 5.1 Time dependence in the field momentum representation

We work in the remainder of this thesis in Euclidean space and in a basis in which the field momentum $\pi = \dot{\phi}$ is diagonal. For the scalar field,

$$\langle\, \pi \,|\hat{\pi}(\mathbf{x}) = \pi(\mathbf{x})\langle\, \pi \,|, \quad i\frac{\delta}{\delta\pi(\mathbf{x})}\langle\, \pi \,| = \langle\, \pi \,|\hat{\phi}(\mathbf{x}). \tag{5.1}$$

We have returned to setting $\hbar = 1$ since all calculations in this chapter will be of the same order. The $\phi$–dependence of the states is made explicit by writing

$$\langle\, \pi \,| = \langle\, N \,| \exp\left( - i \int \mathrm{d}\mathbf{x}\; \hat{\phi}(\mathbf{x})\pi(\mathbf{x})\right), \tag{5.2}$$

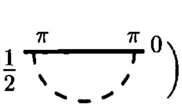
where $\langle\, N \,|$ is the Neumann state annihilated by $\pi$. This representation can be viewed simply as a functional Fourier transform on the space of state functionals,

$$\Psi[\pi, t] := \int \mathcal{D}\phi\; \exp\left(i \int \mathrm{d}^D\mathbf{x}\, \pi(\mathbf{x})\phi(\mathbf{x})\right)\Psi[\phi, t] \tag{5.3}$$

but there is more to learn by working from first principles. The free field vacuum wave functional is

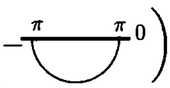
$$\begin{aligned}\Psi_0[\pi] \equiv \langle\, \pi \mid \Psi_0 \rangle &= \langle\, N\, | e^{-i\int \hat{\phi}\pi} e^{-\hat{H}t} |\, \Psi_0\, \rangle \\ &= \int \mathcal{D}\phi \, \exp\Big( - S_E[\phi] - i \int \mathrm{d}^D \phi(\mathbf{x})\pi(\mathbf{x}) \Big)\Big|_{\dot{\phi}(\mathbf{x},0)=0} \\ &= \exp\Big( - \frac{1}{2} \int \mathrm{d}^D(\mathbf{x},\mathbf{y}) \pi(\mathbf{x}) G_{n(0)}(\mathbf{x},0,\mathbf{y},0)\pi(\mathbf{y}) \Big)\end{aligned}$$

where the propagator $G_{n(0)}$ obeys Neumann conditions on the boundary at $t = 0$. A dashed line will represent a propagator with Neumann conditions as the dotted line did for Dirichlet so we write

$$\Psi_0[\pi] = \exp\Big( - \tfrac{1}{2}$$  $$\Big). \qquad (5.4)$$

As in earlier sections, the vacuum wave functional can be reconstructed without the definition above. Computing $G_0$ at equal times as a vacuum expectation value we find

$$G_0(\mathbf{x},0;\mathbf{y},0) = \langle\, \phi(\mathbf{x},0)\phi(\mathbf{y},0)\, \rangle = - \int \mathcal{D}\pi \, \Psi_0[\pi] \frac{\delta}{\delta\pi(\mathbf{x})} \frac{\delta}{\delta\pi(\mathbf{y})} \Psi_0[\pi]$$

$$\Longrightarrow \quad \Psi_0[\pi] = \exp\Big( -$$  $$\Big) \qquad (5.5)$$

and so on beyond the free theory. The two above expressions for the vacuum are equivalent, as can be seen from the method of images,

$$G_{n(0)}(\mathbf{x},t_f,\mathbf{y},t_i) = G_0(\mathbf{x},t_f,\mathbf{y},t_i) + G_0(\mathbf{x},t_f,\mathbf{y},-t_i). \qquad (5.6)$$

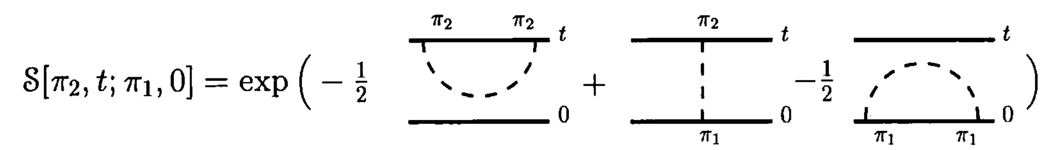
The Schrödinger functional now describes imaginary time evolution. It is defined by an expression analogous to (2.1),

$$\begin{aligned}\mathcal{S}[\pi_2,t;\pi_1,0] &= \langle\, \pi_2\, | e^{-\hat{H}t} |\, \pi_1\, \rangle \\ &= \langle\, N\, | e^{-i\int \pi_2\hat{\phi}} e^{-\hat{H}t} e^{i\int \pi_1\hat{\phi}} |\, N\, \rangle \\ &= \int \mathcal{D}\phi \, \exp\Big( - S_E[\phi] - i \int \mathrm{d}^D \phi(\mathbf{x},t)\pi_2(\mathbf{x}) + i \int \mathrm{d}^D \phi(\mathbf{x},0)\pi_1(\mathbf{x}) \Big)\Big|_{\dot{\phi}(\mathbf{x},0)=0.}^{\dot{\phi}(\mathbf{x},t)=0}\end{aligned} \qquad (5.7)$$

The result of the free field integral is

$$\begin{aligned}\mathcal{S}[\pi_2,t;\pi_2,0] = \exp\Big(-\iint \mathrm{d}^D\mathrm{x}\mathrm{d}^D\mathrm{y}\ &\frac{1}{2}\pi_1(\mathbf{y})G_{\mathrm{orb}}(\mathbf{y},0,\mathbf{x},0))\pi_1(\mathbf{x})\\ &-\pi_2(\mathbf{y})G_{\mathrm{orb}}(\mathbf{y},t,\mathbf{x},0)\pi_1(\mathbf{x})\\ &+\frac{1}{2}\pi_2(\mathbf{y})G_{\mathrm{orb}}(\mathbf{y},t,\mathbf{x},t)\pi_2(\mathbf{x})\Big)\end{aligned} \tag{5.8}$$

where $G_{\mathrm{orb}}$ obeys Neumann, rather than Dirichlet, boundary conditions at $x^0 = t$ and $x^0 = 0$, and there is no longer a time derivative on the propagators since this has been moved into the fields, $\pi = \dot{\phi}$. In terms of Feynman diagrams




(5.9)

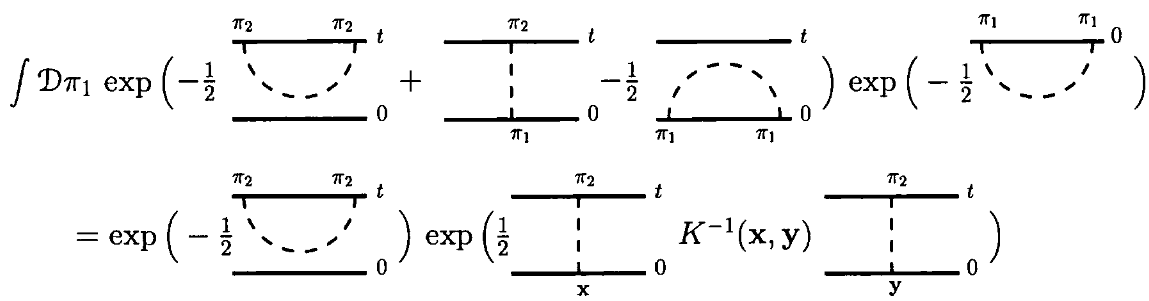
The method of images with the free space propagator gives us the required boundary conditions,

$$G_{\mathrm{orb}}(\mathbf{x}_f,t_f,\mathbf{x}_i,t_i) = \sum_{n\in\mathbb{Z}} G_0(\mathbf{x}_f,t_f,\mathbf{x}_i,t_i+2nt) + G_0(\mathbf{x}_f,t_f,\mathbf{x}_i,-t_i+2nt). \tag{5.10}$$

The sum over images is written as a sum over paths by again compactifying the time direction on the $\mathbb{S}^1/\mathbb{Z}_2$ orbifold of radius $t/\pi$ but without the minus sign weightings of the field representation. Therefore the field momentum Schrödinger functional is constructed from a sum over paths of particles moving on $\mathbb{S}^1/\mathbb{Z}_2 \times \mathbb{R}^D$.

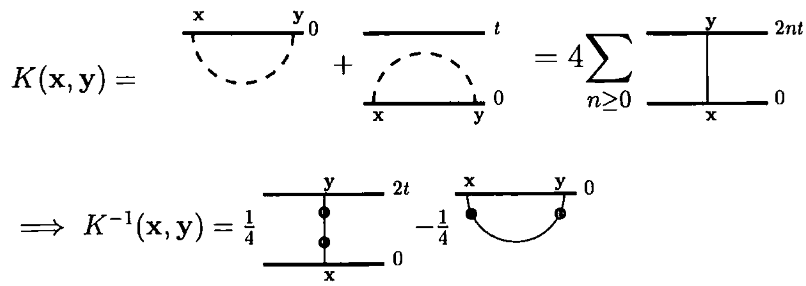
The following calculations are similar to those in Sections 2.3 and 2.5, but are included for completeness. We will demonstrate the correct time evolution of the vacuum and the time dependence of the two point function. The free field integral evolving the vacuum between times 0 and $t$ is

$$\int \mathcal{D}\pi_1 \mathcal{S}[\pi_2,t;\pi_1,0]\,\Psi_0[\pi_1;0] =$$



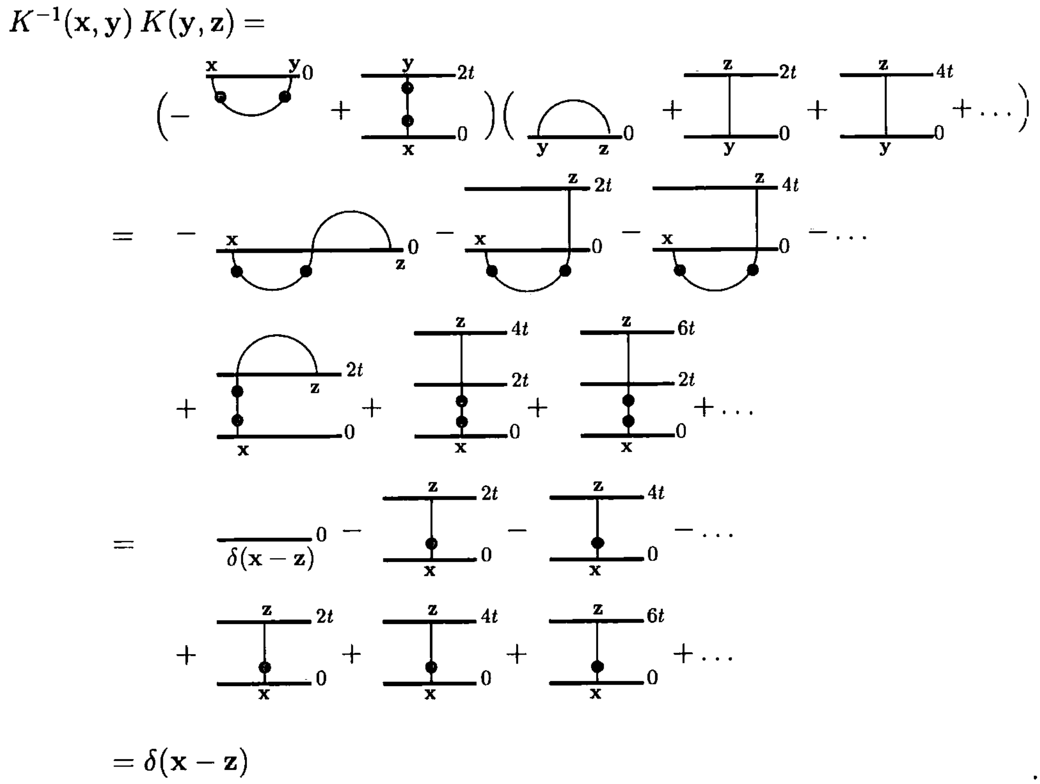

(5.11)

The first three terms are the Schrödinger functional, the final term is the vacuum. This time the operator $K$ and its inverse are given by



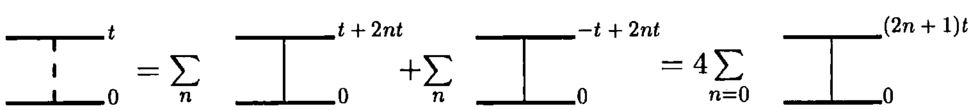
(5.12)

Again we can check that the inverse is correct,



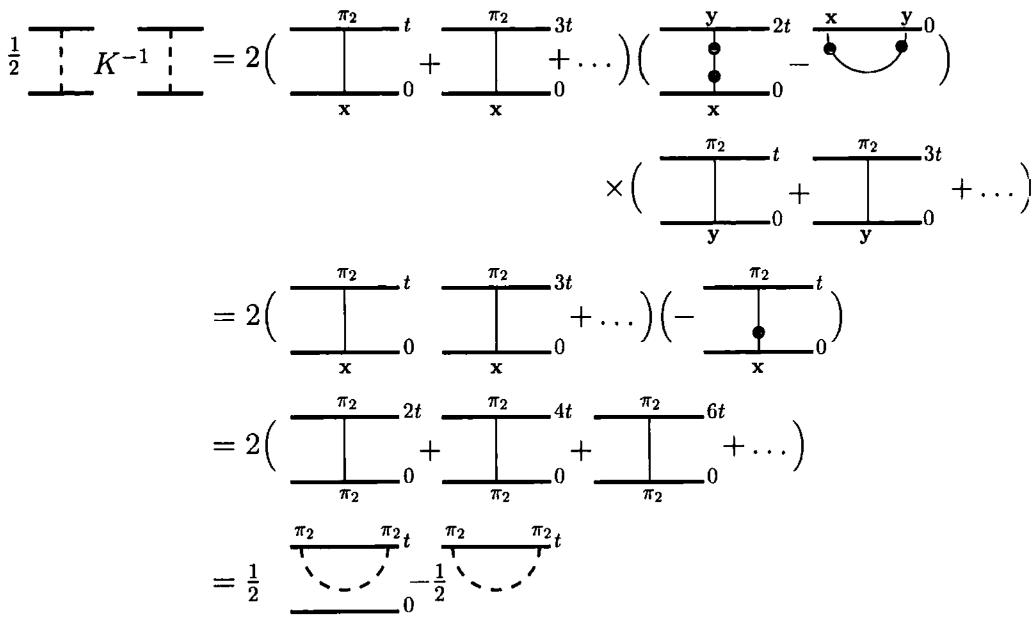
(5.13)

The Schrödinger functional term to be contracted with $K^{-1}$ is



(5.14)

Finally, the Gaussian integral gives



(5.15)

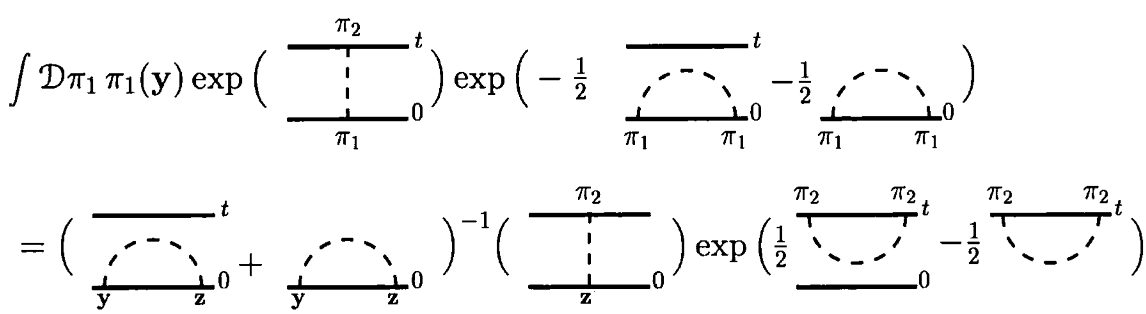
which again implies the correct result for time evolution of the vacuum,

$$\int \mathcal{D}\pi_1 \mathcal{S}[\pi_2, t; \pi_1, 0]\, \Psi_0[\pi_1; 0] = \exp\Big(-\frac{1}{2}$$  $$\Big). \tag{5.16}$$

The two point function in the field momentum representation is

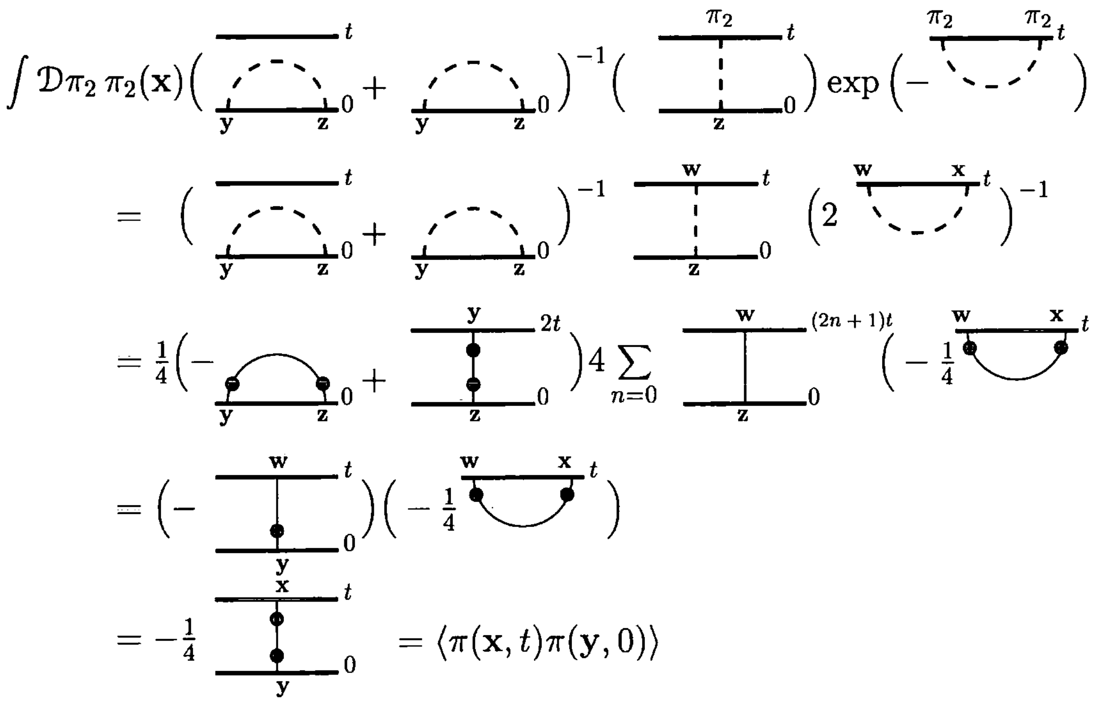
$$\langle \pi(\mathbf{x}, t)\pi(\mathbf{y}, 0)\rangle = \int \mathcal{D}(\pi_2, \pi_1)\Psi_0[\pi_2]\pi_2(\mathbf{x})\mathcal{S}[\pi_2, t; \pi_1, 0]\pi_1(\mathbf{y})\Psi_0[\pi_1]. \tag{5.17}$$

The $\pi_1$ integral is



(5.18)

and the final integral is



(5.19)

In the final line, the dots on the propagator appear since we are correlating the field momenta $\pi = \dot{\varphi}$. The prefactor is a result of our conventions; the minus sign is an artifact of Euclideanisation, as in (2.50), and the quarter cancels the four coming from the time derivatives, giving the correct two point function. This can be verified by using the field momentum vacuum functional to generate the leading term in the equal time two point function,

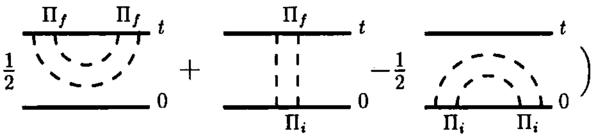
$$\begin{aligned}\langle \pi(\mathbf{x})\pi(\mathbf{y})\rangle &= \int \mathfrak{D}\pi \ \pi(\mathbf{x})\pi(\mathbf{y}) \ \Psi_0[\pi]^2 \\ &= (2G_n(\mathbf{x},0;\mathbf{y},0))^{-1} \\ &= -\tfrac{1}{4}\end{aligned}$$

. (5.20)

## 5.2 T-duality in the closed string Schrödinger functional

The string field Schrödinger functional in the momentum representation is

$$\mathcal{S}[\Pi_f, t; \Pi_i, 0] = \mathcal{N}_{\mathcal{S}} \exp$$




(5.21)

for momentum string fields $\Pi$, and the double line represents the orbifolded propagator for either the open or closed string. The orbifold leads naturally to the question of what rôle T-duality plays. We now show that T-duality exchanges the states attached to the propagators with backgrounds in the dual picture, and vice versa.

The closed string Schrödinger functional is T-dual to the loop diagrams appearing in the normalisation $\mathcal{N}_S$ of the open string Schrödinger functional. We set $t = \pi R$, making the orbifold radius explicit, and use Poisson resummation and a change in modular parameter to convert the closed propagators into open loops. From (5.21) we can write the logarithm of the Schrödinger functional as

$$\log \mathcal{S}_{\text{closed}} = -\frac{1}{2}\Pi_f \sum_{n \text{ even}} G_{\pi R n}\Pi_f + \Pi_f \sum_{n \text{ odd}} G_{\pi R n}\Pi_i - \frac{1}{2}\Pi_i \sum_{n \text{ even}} G_{\pi R n}\Pi_i \qquad (5.22)$$

where $G_t$ is shorthand for the free string propagator with boundary data $X_f^0 - X_i^0 = t$, and the sums are over the images, see (5.12) and (5.14). Applying the Poisson resummation formula

$$\sum_{n \in \mathbb{Z}} e^{-\pi a n^2 + 2\pi i n b} = \frac{1}{\sqrt{a}} \sum_{m \in \mathbb{Z}} e^{-\frac{\pi}{a}(m+b)^2} \qquad (5.23)$$

and changing modular parameter $T \to 1/T$ to represent the closed string propagators as open string loops we can write the Schrödinger functional as

$$-\frac{1}{2} \sum_{n \text{ even}} \Pi_f G_{\pi \widetilde{R} n} \Pi_f + \sum_{n \text{ even}} \Pi_i G_{\pi \widetilde{R} n} e^{in\pi/2} \Pi_f - \frac{1}{2} \sum_{n \text{ even}} \Pi_i G_{\pi \widetilde{R} n} \Pi_i \qquad (5.24)$$

where the sums impose the Neumann boundary conditions on the propagators, $\widetilde{R} = \alpha'/R$ and $G$ is now viewed as an open string loop contribution. The new exponent comes from the resummation formula.

The fields originally glued onto the Dirichlet sections of the closed string propagator become an averaging over backgrounds, characterised by $\Pi_i$ and $\Pi_f$, coupling to the ends of the open string. These backgrounds, as they must be, are the same at each end of the string, for we can write the above as

$$\log \mathbb{S}_{\text{closed}} = -\frac{1}{2}\sum_{n\text{ even}} \left(\Pi_i - e^{iA\int \mathrm{d}X^0}\Pi_f\right) G_{\pi\tilde{R}n}\left(\Pi_i - e^{iA\int \mathrm{d}X^0}\Pi_f\right) \tag{5.25}$$

with Wilson line value $A = (2\widetilde{R})^{-1}$. Let us give an explicit example. We will focus on the co-ordinates. Consider the reparametrisation invariant boundary states

$$\delta^p(\mathbf{X}(\sigma) - \mathbf{q}_{i,f}) \tag{5.26}$$

for $\mathbf{q}_{i,f}$ constant $p$-vectors. These are pointlike states in $p$ directions and Neumann states in $25-p$ directions, $0 \le p \le 25$. The closed string Schrödinger functional is

$$\begin{aligned} \log \mathbb{S}_{\text{closed}} = &- \mathrm{Vol}^{25-p}\int_0^\infty \frac{\mathrm{d}T}{T^{\frac{p+1}{2}}}\, e^{2T} \prod_{m=1}\left(1 - e^{-2mT}\right)^{-24} \sum_{n\text{ even}} e^{-\frac{\pi^2R^2}{2\alpha' T}n^2} \\ &+ \mathrm{Vol}^{25-p}\int_0^\infty \frac{\mathrm{d}T}{T^{\frac{p+1}{2}}}\, e^{-\frac{\delta \mathbf{q}^2}{2\alpha' T}+2T} \prod_{m=1}\left(1 - e^{-2mT}\right)^{-24} \sum_{n\text{ odd}} e^{-\frac{\pi^2R^2}{2\alpha' T}n^2} \end{aligned} \tag{5.27}$$

with $\delta\mathbf{q} = \mathbf{q}_f - \mathbf{q}_i$. Since the open string runs from $\sigma = 0 \dots \pi$ and the closed string from $\sigma = 0 \dots 2\pi$ we must scale the closed string worldsheet to interpret (5.27) as an open loop. We include this in a change of modular parameter $U := 2\pi^2/T$. After this and the Poisson resummation we find

$$\begin{aligned} \log \mathbb{S}_{\text{closed}} = -\mathrm{Vol}^{25-p}\int_0^\infty \frac{\mathrm{d}U}{U}\frac{1}{U^{\frac{26-p}{2}}} e^{U} \prod_{m=1}\left(1 - e^{-mU}\right)^{-24} \\ \times \sum_{n\text{ even}} e^{-\frac{\pi^2\tilde{R}^2}{4\alpha' U}n^2}\left(1 - e^{\frac{in\pi}{2}} e^{-\frac{\delta \mathbf{q}^2 U}{4\pi\alpha'}}\right). \end{aligned} \tag{5.28}$$

Now consider an open string loop. The measure on moduli space is $\mathrm{d}U/U$ (this can be viewed as giving the logarithm of the trace of the worldsheet propagator). If the string has Neumann conditions at its endpoints in $26-p$ directions (including $X^0$) and Dirichlet conditions in $p$ directions, as for a string on a D$(25-p)$-brane then the trace over $\mathbf{X}$ gives the eta function and the factor $(U^{-1/2}\mathrm{Vol})^{(25-p)}$ from the $25-p$ zero modes. The sum and remaining factor of $U^{-1/2}$ come from the trace over $X^0$ in the co-ordinate representation. We arrive at (5.28), if the term in large brackets, coming form the original boundary states represents an averaging over backgrounds of Wilson lines and D$(25-p)$-branes of separation $\delta\mathbf{q}$.

## 5.3 T-duality in the open string Schrödinger functional

We interpret the open string duality as taking us from one Schrödinger functional to another with an exchange of boundary states and backgrounds. Poisson resummation implies

$$\begin{aligned}\left(\frac{\pi R^2}{\alpha' T}\right)^{1/2}\sum_{n\text{ even}} e^{-\frac{\pi^2 R^2}{4\alpha' T}n^2} &= \sum_{n\text{ odd}} e^{-\frac{\bar{R}^2 T}{4\alpha'}n^2} + \sum_{n\text{ even}} e^{-\frac{\bar{R}^2 T}{4\alpha'}n^2}\\ \left(\frac{\pi R^2}{\alpha' T}\right)^{1/2}\sum_{n\text{ odd}} e^{-\frac{\pi^2 R^2}{4\alpha' T}n^2} &= -\sum_{n\text{ odd}} e^{-\frac{\bar{R}^2 T}{4\alpha'}n^2} + \sum_{n\text{ even}} e^{-\frac{\bar{R}^2 T}{4\alpha'}n^2}\end{aligned} \tag{5.29}$$

where the dual radius is now $\overline{R} = 2\alpha'/R$. Following a modular transformation $S := \pi^2/T$ these are the sums in the open string Schrödinger functional. Again the states now represent an averaging over backgrounds. The new momentum states are characterised by the original Neumann condition on the open string ends. The open string Schrödinger functional becomes

$$\log \mathbb{S}_{\text{open}} = -\frac{1}{2}\sum_{n\text{ even}}(\Pi_i - \Pi_f)G_{n\pi\overline{R}}(\Pi_i - \Pi_f) - \frac{1}{2}\sum_{n\text{ odd}}(\Pi_i + \Pi_f)G_{n\pi\overline{R}}(\Pi_i + \Pi_f) \tag{5.30}$$

To interpret this as strings moving in a single background we can introduce a Wilson line, $\Pi_i - e^{iA\int \mathrm{d}X^0}\Pi_f$, with value $A = (\overline{R})^{-1}$.

Explicit examples are difficult to construct for two reasons: difficulties in finding reparametrisation invariant or BRST invariant states, and keeping track of the corner anomaly in the states and backgrounds. A simple example of a state independent of parameterisation would seem to be a string collapsed to a point, as we used for the closed string. The closed string invariant states are more commonly known as Ishibashi states [53] arising via closed string channel descriptions of open string loop amplitudes [54]. They are characterised by the condition

$$L_n - \bar{L}_{-n} = 0 \qquad \forall n \in \mathbb{Z} \tag{5.31}$$

(at worldsheet $\tau = 0$ for simplicity). The pointlike, or localised, state is given by

$$|\,D; q\,\rangle = \exp\left(\sum_{n=1}\frac{1}{n}\alpha_{-n}\cdot\tilde{\alpha}_{-n}\right)|\,q\,\rangle, \qquad \hat{X}^\mu(\sigma,0)|\,D;q\,\rangle = q^\mu|\,D;q\,\rangle$$

and the state $|\, q\,\rangle$ is an eigenstate of the centre of mass of the co-ordinate $X^\mu$. By writing the Virasoro generators in terms of the oscillator modes $\alpha_n$ it is easy to check that this state satisfies (5.31). The corresponding generators for the open string are usually taken to be $L_n - L_{-n}$. This was derived in [55] by inserting the mode expansion for $X^\mu$ into the functional expression for the generators $M_n$ of reparametrisations,

$$M_n := \sqrt{\frac{2}{\pi}} \int_0^\pi \mathrm{d}\sigma \, \sin(n\sigma) X'(\sigma) \frac{\delta}{\delta X(\sigma)}. \tag{5.32}$$

Since this is an operator expression the authors chose to normal–order the result. It is however already well defined – normal ordering unreasonably removes a finite constant. The generators are in fact

$$M_n = \frac{1}{\sqrt{2\pi}} \left( L_n - L_{-n} + \frac{nd}{4} \delta_{n/2 \in \mathbb{Z}} \right) \tag{5.33}$$

in $d$ dimensions. Consistency can be checked in that the Neumann state $|\, N\,\rangle$ obeying $\hat{\pi}(\sigma)|\, N\,\rangle = 0$ is killed by $M_n \, \forall\, n$, as is appropriate from (5.32). Now, the pointlike open string state $|\, D\,\rangle$ obeys

$$\begin{gathered} |\, D; q\,\rangle_{\text{open}} = \exp\left( \sum_{n=1} \frac{1}{2n} \alpha_{-n} \alpha_{-n} \right) |\, q\,\rangle, \\ \left( L_n - L_{-n} - \frac{nd}{4} \delta_{n/2 \in \mathbb{Z}} \right) |\, D; q\,\rangle = 0. \end{gathered} \tag{5.34}$$

The eigenvalue has the wrong sign to be a solution. Similar observations have been made before in the consideration of pointlike states, for example [56]. However, the state is indeed independent of the co-ordinate $\sigma$,

$$\partial_\sigma \hat{X}(\sigma, 0)|\, D; q\,\rangle = 0,$$

and the state wave functional $\langle\, X \mid D; q\rangle$ is reparametrisation invariant – it is a delta functional,

$$\langle\, X \mid D; q\rangle = \delta\left( X(\sigma) - q \right) = \int \mathcal{D}\lambda^\mu \, \exp\left( \left( \int \mathrm{d}\sigma \sqrt{g}\, \lambda^\mu (X_\mu - q_\mu) \right) \right), \tag{5.35}$$

with reparametrisation invariant measure $(\lambda, \lambda) = \int \mathrm{d}^2\xi \sqrt{g} \lambda_\mu \lambda^\mu$. There is a mismatch between the functional and Hilbert space approaches. In the latter the integral over the metric has been performed and so the meaning of a reparametrisation is unclear, to which we attribute the discrepancy. In Chapter 3 the corner anomaly manifests itself in the BRST transformation– recall equation (3.79). BRST transformations correspond to reparameterisations and the corner anomaly indeed explains the additional terms in 5.33, which do not and cannot appear in the closed string generators. If we expand the ghost $c^\sigma$ in modes in equation (3.79),

$$c^\sigma_{\rm b}(\sigma) = \sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty} c_n \sin n\sigma$$
$$\Longrightarrow c^{\sigma\prime}_{\rm b}(0) + c^{\sigma\prime}_{\rm b}(\pi) = \sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty} 2nc_n\, \delta_{n/2\in\mathbb{Z}},$$

then each ghost mode multiplies a generator of reparametrisations $M_n$ and a constant term, so that the Ward identity for BRST invariance becomes

$$\sum_n c_n\big(L_n - L_{-n} + \text{ corner contributions}\big)$$

as found in (5.33).

Let us give an example of the open string duality based on the pointlike and Neumann states. Take the states attached to the Schrödinger functional to be the open string equivalent of (5.26). The logarithm of the Schrödinger functional becomes

$$\begin{aligned}\frac{\log \mathrm{S}}{\mathrm{Vol}^{25-p}} = &-\int_0^\infty \frac{\mathrm{d}T}{T^{\frac{p+1}{2}}}\sum_{n\ \text{even}} e^{-\frac{\pi^2R^2}{4\alpha' T}n^2+T}\prod_{m=1}\left(1-e^{-2mT}\right)^{-12}\\ &+\int_0^\infty \frac{\mathrm{d}T}{T^{\frac{p+1}{2}}} e^{-\frac{(\mathrm{x}_f-\mathrm{x}_i)^2}{4\alpha' T}}\sum_{n\ \text{odd}} e^{-\frac{\pi^2R^2}{4\alpha' T}n^2+T}\prod_{m=1}\left(1-e^{-2mT}\right)^{-12}\end{aligned} \tag{5.36}$$

where 'Vol' is the volume of space generated from attaching Neumann states. Following the Poisson resummations in (5.29) the logarithm of the Schrödinger functional becomes

$$\begin{aligned}\frac{\log \mathrm{S}}{\mathrm{Vol}^{25-p}} = &-\int_0^\infty \frac{\mathrm{d}T}{T^{p/2}}\sum_{n\ \text{even}} e^{-\frac{\tilde{R}^2T}{4\alpha'}n^2+T}\left(1+e^{-\frac{(\mathrm{x}_f-\mathrm{x}_i)^2}{4\alpha' T}}\right)\prod_{m=1}\left(1-e^{-2mT}\right)^{-12}\\ &+\int_0^\infty \frac{\mathrm{d}T}{T^{p/2}}\sum_{n\ \text{odd}} e^{-\frac{\tilde{R}^2T}{4\alpha'}n^2+T}\left(-1-e^{-\frac{(\mathrm{x}_f-\mathrm{x}_i)^2}{4\alpha' T}}\right)\prod_{m=1}\left(1-e^{-2mT}\right)^{-12}\end{aligned} \tag{5.37}$$

The terms inside the large brackets represent the new background for the string. As with the closed string, we could now perform a change of variable $U = \pi^2/T$ which would give us back the sums and determinants of (5.36). Before doing this we describe how to interpret the above expression in terms of a Schrödinger functional constructed in a different gauge. Choosing our gauge fixed worldsheet metric to be $\tilde{g} = \mathrm{diag}(T^2, 1)$ amounts to rotating the worldsheet by 90° so that $T$ represents the length of the string (the co-ordinate ranges are reversed $\sigma \in [0, 1]$, $\tau \in [0, \pi]$). The $X^0$ contributions to the Schrödinger functional in this gauge are

$$e^{T/24} \prod_{m=1} \left(1 - e^{-2mT}\right)^{-1/2} \sum e^{-\frac{\tilde{R}^2 T}{4\alpha'} n^2}.$$

A co-ordinate with Neumann conditions on its endpoints attached to Neumann states gives

$$\mathrm{Vol}\; e^{T/24} \prod_{m=1} \left(1 - e^{-2mT}\right)^{-1/2}$$

and a co-ordinate with Dirichlet conditions on its endpoints attached to a Neumann state gives

$$\frac{1}{T^{1/2}} e^{T/24} \prod_{m=1} \left(1 - e^{-2mT}\right)^{-1/2}$$

in addition to any contribution from the background it couples to. We conclude that (5.37) is the Schrödinger functional for a string in this gauge, with initial and final field momentum Neumann states $\Pi = 1$, with an averaging over backgrounds of D(25-p)-branes and Wilson lines.

We might expect to see a description of the same system if we perform the change of variables $U = \pi^2/T$ on the modular parameter. Doing this gives us

$$\begin{aligned}\frac{\log \mathcal{S}}{\mathrm{Vol}^{25-p}} = &- \int_0^\infty \frac{\mathrm{d}U}{U^{\frac{26-p}{2}}} U^5 \sum_{n \text{ even}} e^{-\frac{\tilde{R}^2 \pi^2}{4\alpha' U} n^2 + U} \left(1 + e^{-\frac{U(x_f - x_i)^2}{4\alpha' \pi^2}}\right) \prod_{m=1} \left(1 - e^{-2mU}\right)^{-12} \\ &+ \int_0^\infty \frac{\mathrm{d}U}{U^{\frac{26-p}{2}}} U^5 \sum_{n \text{ odd}} e^{-\frac{\tilde{R}^2 \pi^2}{4\alpha' U} n^2 + U} \left(-1 - e^{-\frac{U(x_f - x_i)^2}{4\alpha' \pi^2}}\right) \prod_{m=1} \left(1 - e^{-2mU}\right)^{-12}\end{aligned} \tag{5.38}$$

There is an additional factor of $U^5$ in both terms as compared to the 'expected' result. However, we have not been careful with the corner anomaly. The change of modular parameter $T \to \pi^2/T$ corresponds to a scaling of the metric. To see this it is helpful to put the $\sigma$ parameter range into the metric so $\hat{g}_{\sigma\sigma} \to \pi^2$ then the change of variable corresponds to a scaling,

$$g_{ab} = \begin{pmatrix} \pi^2 & 0 \\ 0 & T^2 \end{pmatrix} \to \begin{pmatrix} \pi^4/T^2 & 0 \\ 0 & \pi^2 \end{pmatrix} = \frac{\pi^2}{T^2} \begin{pmatrix} \pi^2 & 0 \\ 0 & T^2 \end{pmatrix}. \tag{5.39}$$

This is a constant Weyl scaling with Liouville mode $\rho = 2\log(\pi^2/T) = 2\log U$. The logarithms of the various determinants in the Polyakov integral depend linearly on $\rho$ at the corners, so a scaling must produce powers of $U$ in the propagator. Joining states and background contributions produces the same effect, thus the additional factors in (5.38).

## 5.4 Conclusions and outlook

We have shown that the scalar field Shrödinger functional can be written in terms of particles moving on $\mathbb{R}^D \times \mathbb{S}^1/\mathbb{Z}_2$, and that the action of the Schrödinger functional, describing time evolution, reduces to a Feynman diagram expansion and the gluing property.

The bosonic string field propagator, for both the open and closed string, obeys a generalisation of the gluing property which sews together the propagator worldsheets at definite times. The string field Schrödinger functional can therefore be written down using the diagram expansion found in the field theory case. Although some Plank scale effects due to the extension of the string may have been expected to spoil our notions of time, our results seem to imply that we may regard time in string field theory just as we do in a particle theory.

The string field gluing property, unlike the scalar field case, depends on treating the gauge fixing ghosts (or alternatively the metric) at least on an equal level as the co-ordinates. While this is not surprising, it may have some implications for the nature of causality, further study is required.

Despite being interested in time dependence, we have not made an issue of the faster than light tachyon in the string spectrum. It appears as a divergence in the string field propagator which is to be regulated. We have, implicitly, replaced the mass squared by a small positive parameter and returned it to its proper value at the end of calculations. We believe this is justified since we could repeat our calculations (which would be lengthier but no more difficult) using a superstring field theory based on the conformal action of Ramond-Neveu-Schwarz which would yield the same physics but without the tachyon divergence.

Timelike T-duality of the string theory becomes a large/small time duality of the string field theory under which boundary states are exchanged with backgrounds coupling to the ends of the dual string. In the examples given we have seen the usual features of T-duality appearing, namely Wilson lines and D-branes.

For the open string, the presence of the corner Weyl anomaly is necessary in any string field theory for building more complicated surfaces by sewing propagators and vertices together and care must be taken to include its effects in calculations; in particular we have shown that the corner anomaly is responsible for old confusions regarding the nature of point-like strings.

The vacuum of string field theory is currently a topic of active research following Sen's conjecture [26] that the tachyon instability comes from a space filling D-25 brane to which the ends of an open string couple, and that this will be seen in measuring the vacuum energy of Witten's string field. We have described how to construct the string field vacuum functional with no assumption of the type of interaction vertex. Although the ghost sector of our theory differs from Witten's we hope that our result may be useful in investigating this area.

Our time co-ordinate, unlike in other string field theories, is a time at which the whole spatially extended string (with ghosts) exists. This seems to work around the problems of, for example, quantising Witten's theory [24] where time is normally taken to be the midpoint of $X^0(\sigma)$, but the string remains extended in time. This may be a worthwhile avenue for future study. As discussed in Section 4.3, we may also postulate an interaction local in time but with arbitrary spatial dependence and repeat our field theory arguments, but we have shown that our methods are capable of handling interaction theories almost 'without' an interaction vertex.

As a very first step to performing a similar construction of string field objects in non-flat spacetimes, we have generalised some of our field theory gluing properties to anti-de Sitter spacetimes of arbitrary dimension. This can be found in the appendix.

The non-perturbative correspondence between string theory in anti-de Sitter space and conformal field theory on the spacetime boundary has attracted much interest in recent years. It has been shown that the AdS/CFT correspondence holds between scalar field theory in $\mathrm{AdS}_{D+1}$ and conformal field theory on $\mathbb{R}^D$ [57]. Our results allow us to describe fundamental properties of the radial evolution of scalar fields in $\mathrm{AdS}_{D+1}$. This is especially interesting as it is related to the 'extra' dimension as seen from the boundary. There is an opportunity to study how our results relate to the holographic description of radial translation as scaling of operators in the boundary conformal field theory.

The gluing property in flat space, with respect to the time direction, determined the field theory time evolution operator and generalised to bosonic strings and string fields. Although we have not attempted it here, it would be intriguing to see if our results could be extended to strings in anti-de Sitter spacetime and to learn how they arise in the dual field theory. A framework in which to begin this task has been set up by Berkovits [58], using pure spinors to overcome the difficulties of covariantly quantising the Green-Schwarz string.

# Appendix A

# Radial evolution in anti-de Sitter spacetime

This material originally appeared in [60]. Since its proposal [12] there has been huge interest in studying the correspondence between string theory in anti-de Sitter space and its holographic dual conformal field theory on the boundary (see [61] for a comprehensive review). A variety of new approaches to studying the correspondence have recently been employed (see for example [58] and [62]... [64]). Berkovits' approach [65] gives a quantisable action for the superstring in AdS space times a compact manifold, with Ramond Ramond flux, but the complexity of the action makes hard work of calculating scattering amplitudes [66]... [69].

Given the difficulties of string calculations in anti-de Sitter space, it is worthwhile to pursue unconventional approaches. Here we show that in anti-de Sitter spacetimes of arbitrary dimension, factorisation of the quantum mechanical path integral agrees with the gluing property of the scalar field propagator which we expect from the second quantised derivation above. This defines the evolution of scalar fields in the radial direction, much as we found time evolution in flat space. Time translation invariance simplified the flat space calculations so we do not expect our Feynman diagram arguments to be so simple in AdS, but we can derive the essential property.

We will represent Euclideanised anti-de Sitter spacetime as the upper half space

$x^0 > 0$ with metric

$$\begin{aligned} ds^2 &= \frac{1}{x^{0\,2}}\left(dx^0 dx^0 + d\mathbf{x}^i d\mathbf{x}^i\right) =: A_{ab} dx^a dx^b, \\ A(x^0) &:= \mathrm{Det}\, A_{ab} = \left(\frac{1}{x^0}\right)^{D+1} \end{aligned} \tag{A.0.1}$$

where $i = 1 \ldots D$ and $x^0$ is the 'radial' direction. For a scalar field of mass $m$ define $\nu = \sqrt{D^2/4 + m^2}$, then the scalar field propagator which vanishes on the boundary of the spacetime (compactified $\mathbb{R}^D$) is, labelling the final and initial values of $x^0$ as $r_f$ and $r_i$ respectively [70] [71],

$$G_{\mathrm{AdS}}(r_f, \mathbf{x}_f; r_i, \mathbf{x}_i) = -\int \frac{d^D\mathbf{k}}{(2\pi)^D} (r_f r_i)^{D/2} e^{-i\mathbf{k}.(\mathbf{x}_f - \mathbf{x}_i)} K_\nu(|\mathbf{k}|r_f) I_\nu(|\mathbf{k}|r_i) \tag{A.0.2}$$

when $r_f > r_i$ and the Bessel functions $I_\nu$ and $K_\nu$ [72] are exchanged otherwise. As before, we insert into the sum over paths in anti-de Sitter space a resolution of the identity,

$$1 = \int d\tau' \, \delta\left(x^0(\tau') - r\right) \left.\frac{dx^0(\tau')}{d\tau'}\right|_{x^0 = r}.$$

After integrating out the reparametrisations we can write the momentum conjugate to the radial variable as

$$\left.\frac{dx_0}{d\tau}\right|_{x^0 = r} = -r^2 \frac{\partial}{\partial r} \tag{A.0.3}$$

which implies that the propagator factorises as

$$\begin{aligned} G_{\mathrm{AdS}}(r_f, \mathbf{x}_f; r_i, \mathbf{x}_i) = \int d^D\mathbf{y} A(r) \, &\left(r^2 \frac{\partial}{\partial r} G_{\mathrm{AdS}}(r_f, \mathbf{x}_f; r, \mathbf{y})\right) G_{\mathrm{AdS}}(r, \mathbf{y}; r_i, \mathbf{x}_i) \\ &- G_{\mathrm{AdS}}(r_f, \mathbf{x}_f; r, \mathbf{y}) \left(r^2 \frac{\partial}{\partial r} G_{\mathrm{AdS}}(r, \mathbf{y}; r_i, \mathbf{x}_i)\right). \end{aligned} \tag{A.0.4}$$

Let $r_f > r > r_i$, then inserting the explicit representation of the propagator (A.0.2) into the right hand side of (A.0.4) we find

$$\begin{aligned} -\int \frac{d^D(\mathbf{k}, \mathbf{p}, \mathbf{y})}{(2\pi)^{2D}} &(r_f r_i)^{D/2} \, e^{-i\mathbf{k}.(\mathbf{x}_f - \mathbf{y})} e^{-i\mathbf{p}.(\mathbf{y} - \mathbf{x}_i)} K_\nu(|\mathbf{k}|r_f) I_\nu(|\mathbf{p}|r_i) \\ &\times r^{1-D/2} \left\{ K_\nu(|\mathbf{p}|r) \frac{\partial}{\partial r} r^{D/2} I_\nu(|\mathbf{k}|r) - I_\nu(|\mathbf{k}|r) \frac{\partial}{\partial r} r^{D/2} K_\nu(|\mathbf{p}|r) \right\}. \end{aligned} \tag{A.0.5}$$

The integral over $\mathbf{y}$ gives a momentum conserving delta function which allows us to do the integral over $\mathbf{p}$, say. For brevity write $z \equiv |\mathbf{k}|r$ and the result of these integrations is

$$\begin{aligned}
&-\int \frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D}(r_f r_i)^{D/2}\ e^{-i\mathbf{k}.(\mathbf{x}_f-\mathbf{x}_i)}K_\nu(|\mathbf{k}|r_f)I_\nu(|\mathbf{k}|r_i) \\
&\qquad\qquad \times z^{1-D/2}\left\{K_\nu(z)\frac{\partial}{\partial z}z^{D/2}I_\nu(z) - I_\nu(z)\frac{\partial}{\partial z}z^{D/2}K_\nu(z)\right\} \\
&= -\int \frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D}(r_f r_i)^{D/2}\ e^{-i\mathbf{k}.(\mathbf{x}_f-\mathbf{x}_i)}K_\nu(|\mathbf{k}|r_f)I_\nu(|\mathbf{k}|r_i) \\
&\qquad\qquad \times z^{1-\nu}\left\{K_\nu(z)\frac{\partial}{\partial z}z^{\nu}I_\nu(z) - I_\nu(z)\frac{\partial}{\partial z}z^{\nu}K_\nu(z)\right\}
\end{aligned} \tag{A.0.6}$$

plus two terms coming from the derivatives of $z^{D/2}$ which cancel. We require the final line of the above to be unity in order to recover $G_{\mathrm{AdS}}$. Applying the Bessel function properties [72]

$$\frac{\partial}{\partial z}z^\nu I_\nu(z) = z^\nu I_{\nu-1}(z), \qquad \frac{\partial}{\partial z}z^\nu K_\nu(z) = -z^\nu K_{\nu-1}(z), \tag{A.0.7}$$

the final line of (A.0.6) becomes

$$zI_{\nu-1}(z)K_\nu(z) + zI_\nu(z)K_{\nu-1}(z) = 1, \tag{A.0.8}$$

a standard Bessel function identity for unity, leaving

$$-\int \frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D}(r_f r_i)^{D/2}\ e^{-i\mathbf{k}.(\mathbf{x}_f-\mathbf{x}_i)}K_\nu(|\mathbf{k}|r_f)I_\nu(|\mathbf{k}|r_i) \equiv G_{\mathrm{AdS}}(r_f,\mathbf{x}_f;r_i,\mathbf{x}_i). \tag{A.0.9}$$

We have proven (A.0.4), which agrees with the general case given at the beginning of this appendix. For $r_f < r < r_i$ the right hand side of (A.0.4) picks up a minus sign. These results also hold in the limit in which $r_f = r$ or $r = r_i$.

In the AdS/CFT correspondence translations in the bulk radial direction correspond to conformal transformations in the boundary field theory. Applied to the toy model of scalar field theory in $AdS_{D+1}$ and conformal field theory on $\mathbb{R}^D$ [57], there is an opportunity to study how our results relate to a holographic description in terms of scaling of operators in the boundary conformal field theory.

Toward this end we note that our gluing properties hold for the propagator

$$\begin{aligned}
G_\epsilon(r_f,\mathbf{x}_f;r_i,\mathbf{x}_i) &:= G_{\mathrm{AdS}}(r_f,\mathbf{x}_f;r_i,\mathbf{x}_i) \\
&\quad + \int \frac{\mathrm{d}^D\mathbf{k}}{(2\pi)^D}(r_f r_i)^{D/2}e^{-i\mathbf{k}.(\mathbf{x}_f-\mathbf{x}_i)}K_\nu(|\mathbf{k}|r_f)K_\nu(|\mathbf{k}|r_i)\frac{I_\nu(|\mathbf{k}|\epsilon)}{K_\nu(|\mathbf{k}|\epsilon)}
\end{aligned} \tag{A.0.10}$$

considered in [71] which vanishes not on the spacetime boundary but on the near-boundary surface $x^0 = \epsilon$ and is used to regulate boundary divergences when investigating the conformal field theory.

# Appendix B

# Corollaries for sewing strings

Following the derivation of the gluing property for string field theory, we prove here the corollaries we found for quantum field theory, (2.49) (2.50), which are necessary to carry out our diagrammatic calculations. Essentially, the corollaries hold because the time dependence of the string field and quantum field propagators is the same. Recall that the string field propagator can be written

$$G(t_f, \mathbf{B}_f; t_i, \mathbf{B}_i) = \int_0^\infty \frac{\mathrm{d}S}{\sqrt{4\pi S}} e^{-\frac{1}{4S}(t_f - t_i)^2} \langle\, \mathbf{B}_f \,|e^{-\mathbf{H}S}|\, \mathbf{B}_i \,\rangle. \tag{B.0.1}$$

Identify a complete set of eigenvectors $|\, n\,\rangle$ of the Hamiltonian $\mathbf{H}$ and insert this into the given expression,

$$\begin{aligned} G(t_f, \mathbf{B}_f; t_i, \mathbf{B}_i) &= \sum_n \langle\, \mathbf{B}_f \,|\, n \rangle \langle\, n \,|\, \mathbf{B}_i \rangle \int_0^\infty \frac{\mathrm{d}S}{\sqrt{4\pi S}} e^{-\frac{1}{4S}(t_f - t_i)^2 - \lambda_n S} \\ &= \sum_n \langle\, \mathbf{B}_f \,|\, n \rangle \langle\, n \,|\, \mathbf{B}_i \rangle \frac{1}{2\sqrt{\lambda_n}}\; e^{-\sqrt{\lambda_n}|t_f - t_i|} \end{aligned} \tag{B.0.2}$$

This description is analogous to the Fourier representation used for the scalar field propagator, where $\lambda_n$ is $E_n(\mathbf{k})^2$ and the complete set is a $D$-dimensional momentum integral over $\mathbf{k}$. The time dependence of the string field propagator is therefore the same as that of the scalar field propagator. Since we now know that integrating over shared $\langle\, \mathbf{B} \,|$ data when we glue two propagators is a resolution of the identity, the cases (2.22) now follow,

$$\begin{aligned}
&\int \mathcal{D}\mathbf{B}\ G(t_2, \mathbf{B}_2; t, \mathbf{B})\Big( -2\frac{\partial}{\partial t}\Big) G(t, \mathbf{B}; t_1, \mathbf{B}_1) \\
&\qquad = \mathrm{Sg}(t - t_i) \sum_n \langle\, \mathbf{B}_2 \,|\, n \rangle \langle\, n \,|\, \mathbf{B}_1 \rangle \frac{1}{2\sqrt{\lambda_n}}\ e^{-\sqrt{\lambda_n}(|t_f - t| + |t - t_i|)}.
\end{aligned} \tag{B.0.3}$$

The propagator at equal time is

$$G(0, \mathbf{B}_2; 0, \mathbf{B}_1) = \sum_n \langle\, \mathbf{B}_2 \,|\, n \rangle \langle\, n \,|\, \mathbf{B}_1 \rangle \frac{1}{2\sqrt{\lambda_n}}, \tag{B.0.4}$$

and the derivative with respect to both time arguments of the propagator at equal time is

$$G(\dot{0}, \mathbf{B}_2; \dot{0}, \mathbf{B}_1) = \sum_n \langle\, \mathbf{B}_2 \,|\, n \rangle \langle\, n \,|\, \mathbf{B}_1 \rangle \frac{\sqrt{\lambda_n}}{2}, \tag{B.0.5}$$

from which (2.50) follows,

$$\begin{aligned}
&4\int \mathcal{D}\mathbf{B}\ G(0, \mathbf{B}_2; 0, \mathbf{B}) G(\dot{0}, \mathbf{B}_2; \dot{0}, \mathbf{B}) \\
&\qquad = \sum_{n,m} \int \mathcal{D}\mathbf{B}\ \langle \mathbf{B}_2 \,|\, n \rangle \langle\, n \,|\, \mathbf{B} \rangle \langle\, \mathbf{B} \,|\, m \rangle \langle\, m \,|\, \mathbf{B}_i \rangle \frac{\sqrt{\lambda_n}}{\sqrt{\lambda_m}} \\
&\qquad = \sum_{n,m} \langle\, \mathbf{B}_2 \,|\, n \rangle \langle\, n \,|\, m \rangle \langle\, m \,|\, \mathbf{B}_i \rangle \frac{\sqrt{\lambda_n}}{\sqrt{\lambda_m}} \\
&\qquad = \langle\, \mathbf{B}_2 \,|\, \mathbf{B}_1 \rangle \\
&\qquad = \delta\big[\mathbf{B}_2 - \mathbf{B}_1\big].
\end{aligned} \tag{B.0.6}$$

It is a simple matter to repeat the arguments above in Euclidean space, to show that the string field generalisations of (2.21) and (2.22) hold.

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